GLOBAL TORELLI THEOREM FOR PROJECTIVE MANIFOLDS OF CALABI-YAU TYPE

XIAOJING CHEN, FENG GUAN, AND KEFENG LIU

Contents

1. Introduction

In this paper we prove two main results about the global properties of the period maps from the Teichmüller space of polarized and marked Calabi-Yau type manifolds to the period domain of polarized Hodge structures.

Let $(M, L, \{\gamma_1, \cdots, \gamma_m\})$ be a polarized and marked Calabi-Yau type manifold of complex dimension n as defined in Section 2.2, which includes all simply connected Calabi-Yau manifolds and certain Fano manifolds. Here L denotes an ample line bundle on M and $\{\gamma_1, \cdots, \gamma_m)\}$ is a basis of $H^n(M, \mathbb{Z})/\mathrm{Tor}$. The moduli space $\mathcal{M}$ of polarized complex structures on a given compact differentiable manifold X is a complex analytic space consisting of biholomorphically equivalent pairs of complex structure and an ample line bundle (M, L). We use $[M, L]$ to denote the point in $\mathcal{M}$ corresponding to the complex structure on a complex manifold M that is diffeomorphic to X. The corresponding Teichmüller sapce $\mathcal{T}$ can be considered as the moduli space of polarized and marked triples $(M, L, \{\gamma_1, \cdots, \gamma_m\})$, of which we refer to Section 2.3 for details.

For simplicity, in this paper we assume the Teichmüller space $\mathcal{T}$ of polarized and marked Calabi-Yau type manifold $(M, L, \{\gamma_1, \cdots, \gamma_m\})$ is a connected smooth complex manifold together with a versal family

$$\pi : \mathcal{U} \to \mathcal{T}$$

of polarized and marked Calabi-Yau type manifolds, which contains $(M, L, \{\gamma_1, \cdots, \gamma_m\})$ as a fiber. In Section 2, one will see that the Teichmüller space is actually simply connected. We will simply use M to denote the polarized and marked Calabi-Yau type manifold $(M, L, \{\gamma_1, \cdots, \gamma_m\})$.

Let D denote the period domain of the Hodge structures of polarized and marked Calabi-Yau type manifolds of weight n, and $\Phi : \mathcal{T} \to D$ denote the period map from the Teichmüller space of polarized and marked Calabi-Yau type manifolds to the period domain.

By using the fact that the restriction of the versal family of the polarized and marked Calabi-Yau type manifolds on any small coordinate chart of $\mathcal{T}$ is a local Kuranishi family, one gets a local holomorphic coordinate chart around each point in $\mathcal{T}$. From this we construct a global holomorphic affine structure on the Teichmüller space $\mathcal{T}$, which naturally induces a global holomorphic affine connection on $\mathcal{T}$. We then use the global holomorphic affine connection on the Teichmüller space to construct the global holomorphic coordinate functions on $\mathcal{T}$. This construction is given in Theorem 4.2. From Theorem 4.2, we derive the following global Torelli theorem on the Teichmüller space of polarized and marked Calabi-Yau type manifolds:

Theorem 1.1. *Let $\mathcal{T}$ be the Teichmüller space of polarized and marked Calabi-Yau type manifolds, then the period map*

$$\Phi : \mathcal{T} \to D$$

is injective.

If we further assume that the moduli space $\mathcal{M}$ is smooth, and consider the period map $\Phi_{\mathcal{M}} : \mathcal{M} \to D/\Gamma$, where Γ denotes the global monodromy group which acts properly and discontinuously on the period domain D. For simplicity we also assume that the action of Γ is free so that D/Γ is a smooth analytic manifold. The following main theorem of this paper proves that for polarized Calabi-Yau type manifolds the period map $\Phi_{\mathcal{M}}$ is a covering map onto its image.

The global Torelli problem on the moduli space $\mathcal{M}$ asks when $\Phi_{\mathcal{M}}$ is injective, and the generic Torelli problem asks when there exists an open dense subset $U \subset \mathcal{M}$ such that $\Phi_{\mathcal{M}}|_U$ is injective. In general, generic Torelli theorem on the moduli space is weaker than global Torelli theorem on the moduli space $\mathcal{M}$. However as a consequence of this main theorem, we show that if the generic Torelli theorem on the moduli space of polarized Calabi-Yau type manifolds is known, then we also get the global Torelli theorem on the moduli space of polarized Calabi-Yau type manifolds.

Theorem 1.2. *Let $\mathcal{M}$ be the moduli space of polarized Calabi-Yau type manifolds. If $\mathcal{M}$ is smooth, and Γ acts on D freely, then the period map on the moduli space $\Phi_{\mathcal{M}} : \mathcal{M} \to D/\Gamma$ is a covering map from $\mathcal{M}$ onto its image in D/Γ. As a consequence, if the period map on the moduli space $\Phi_{\mathcal{M}}$ is generically injective, then it is globally injective.*

Note that in the most interesting cases it is always possible to find a subgroup Γ_0 of Γ, which is of finite index in Γ, such that Γ_0 acts on freely and thus D/Γ_0 is smooth. In such cases we can consider the lift $\Phi_{\mathcal{M}_0} : \mathcal{M}_0 \to D/\Gamma_0$ of the period map $\Phi_{\mathcal{M}}$, with $\mathcal{M}_0$ a finite cover of $\mathcal{M}$. Then our argument can be applied to prove that $\Phi_{\mathcal{M}_0}$ is actually a covering map onto its image for polarized Calabi-Yau type manifolds with smooth moduli spaces.

This paper is organized as follows: in Section 2, we review the definition of the classifying space of polarized Hodge structures, give the definition of the Calabi-Yau type manifolds, and describe definition of the moduli space of polarized Calabi-Yau type manifolds and the Teichmüller space of polarized and marked Calabi-Yau type manifolds, together with our basic assumptions on the Teichmüller space and its properties. Then we introduce the period map from the Teichmüller space of polarized and marked Calabi-Yau type manifolds to the period domain.

In Section 3, we review the concepts of affine manifolds and affine maps following Matsushima [13], Vitter [25], and Auslander and Markus [2]. Based on the local Kuranishi deformation theory, we construct a local holomorphic coordinate chart around each point in the Teichmüller space $\mathcal{T}$, and this gives the Kuranishi coordinate cover on $\mathcal{T}$. We then show that this Kuranishi coordinate cover gives a holomorphic affine structure on the Teichmüller space $\mathcal{T}$ of polarized and marked Calaib-Yau type manifolds.

In Section 4, we use the holomorphic affine structure constructed in Section 3 and the property that $\mathcal{T}$ is simply connected to prove the existence of a global holomorphic flat coordinate, and prove the holomorphic affine embedding from $\mathcal{T}$ into $\mathbb{C}^N \cong H_p^{s-1,n-s+1}$. We then use this global holomorphic affine coordinate and the affine embedding to prove the global Torelli theorem on the Teichmüller space of polarized and marked Calabi-Yau type manifolds.

In Section 5, following [17] we describe a more geometric way to understand the holomorphic affine structure on the Teichmüller space of polarized and marked Calabi-Yau type manifolds by using the orbit N_+ in D of the unipotent group. We define a Hodge completion $\overline{\mathcal{T}}$ of the Teichmüller space of polarized and marked Calabi-Yau type manifolds, and extend the period map to $\check{\Phi} : \overline{\mathcal{T}} \to N_+ \hookrightarrow \check{D}$. At the end of the paper, we make smoothness assumptions on both the moduli space $\mathcal{M}$ of polarized Calabi-Yau type manifolds and on the quotient space D/Γ, then use the global Torelli theorem on the Teichmüller space of polarized and marked Calabi-Yau type manifolds, together with the geometry of the unipotent orbit, we show that the period map $\Phi_{\mathcal{M}} : \mathcal{M} \to D/\Gamma$ is a covering map onto its image, and conclude that generic Torelli implies global Torelli on the moduli space.

Acknowledgement: We would like to thank Professors Wilfried Schmid, Andrey Todorov and Shing-Tung Yau for sharing many of their ideas.

2. The period map on the Teichmüller space

In Section 2.1, we review the construction of the classifying space of polarized Hodge structures and its basic properties. We refer the reader to Section 2 and Section 3 of [16] for more details. In Section 2.2, we introduce the definition of the polarized and marked Calabi-Yau type manifolds and the basic properties. In Section 2.3, we briefly describe the definitions of the moduli space of polarized Calabi-Yau type manifolds and

the Teichmüller space of polarized and marked Calabi-Yau type manifolds. We then make basic assumptions on the Teichmüller space, and show the simply connectedness property on the Teichmüller space. In Section 2.4 we introduce the period map from the Teichmüller space of the polarized marked Calabi-Yau type manifolds to the period domain.

2.1. **Classifying space of polarized Hodge structures.** Let $H_{\mathbb{R}}$ be a real vector space of dimension m with a $\mathbb{Z}$-structure defined by a lattice $H_{\mathbb{Z}} \subset H_{\mathbb{R}}$, and let $H_{\mathbb{C}}$ be the complexification of $H_{\mathbb{R}}$. A Hodge structure of weight n on $H_{\mathbb{C}}$ is a decomposition

$$H_{\mathbb{C}} = \bigoplus_{k=0}^{n} H^{k,n-k}, \quad \text{with} \quad H^{n-k,k} = \overline{H^{k,n-k}}.$$

The integers $h^{k,n-k} = \dim_{\mathbb{C}} H^{k,n-k}$ are called the Hodge numbers. To each Hodge structure of weight n on $H_{\mathbb{C}}$, one assigns the Hodge filtration:

$$H_{\mathbb{C}} = F^0 \supset \cdots \supset F^n, \tag{1}$$

with $F^k = H^{n,0} \oplus \cdots \oplus H^{k,n-k}$, and $f^k = \dim_{\mathbb{C}} F^k = \sum_{i=k}^{n} h^{i,n-i}$. This filtration has the property that

$$H_{\mathbb{C}} = F^k \oplus \overline{F^{n-k+1}}, \quad \text{for } 0 \leq k \leq n. \tag{2}$$

Conversely, every decreasing filtration (1), with the property (2) and fixed dimensions $\dim_{\mathbb{C}} F^k = f^k$, determines a Hodge structure $\{H^{k,n-k}\}_{k=0}^{n}$, with

$$H^{k,n-k} = F^k \cap \overline{F^{n-k}}.$$

A polarization for a Hodge structure of weight n consists of the data of a Hodge Riemann bilinear form Q over $\mathbb{Z}$, which is symmetric for even n, skew symetric for odd n, such that

$$Q(H^{k,n-k}, H^{r,n-r}) \ = 0 \ \text{ unless } \ k = n - r, \tag{3}$$

$$i^{2k-n} Q(v, \overline{v}) \ > 0 \ \text{ if } v \in H^{k,n-k}, v \neq 0. \tag{4}$$

In terms of the Hodge filtration $F^n \subset \cdots \subset F^0 = H_{\mathbb{C}}$, the relations (3) and (4) can be written as

$$Q\left(F^k, F^{n-k+1}\right) = 0, \tag{5}$$

$$Q\left(Cv, \overline{v}\right) > 0 \ \text{ if } \ v \neq 0, \tag{6}$$

where C is the Weil operator given by $Cv = i^{2k-n} v$ when $v \in H^{k,n-k}$.

The classifying space or the period domain D for polarized Hodge structures with Hodge numbers $\{h^{k,n-k}\}_{k=0}^{n}$ is the space of all such Hodge filtrations

$$D = \left\{F^n \subset \cdots \subset F^0 = H_{\mathbb{C}} \mid \dim F^k = f^k, (5) \text{ and } (6) \text{ hold}\right\}.$$

The compact dual $\check{D}$ of D is

$$\check{D} = \left\{F^n \subset \cdots \subset F^0 = H_{\mathbb{C}} \mid \dim F^k = f^k \text{ and } (5) \text{ hold}\right\}.$$

The classifying space or the period domain $D \subset \check{D}$ is an open subset.

From the definition of classifying space we naturally get the Hodge bundles over $\check{D}$ by associating to each point in $\check{D}$ the vector spaces $\{F^k\}_{k=0}^{n}$ in the Hodge filtration of that point. Without confusion we will also denote by F^k the bundle with F^k as the fiber, for

each $0 \le k \le n$. In the rest of this paper, we may simply use Hodge structure to mean polarized Hodge structure of weight n.

Let $\{F^n \subset \cdots \subset F^0\}$ be a point in D, then the tangent space of D at this point is $\bigoplus_{k=0}^{n} \mathrm{Hom}(F^k, H_{\mathbb{C}})$. We are interested in the horizontal tangent space $\bigoplus_{k=0}^{n} \mathrm{Hom}(F^k, F^{k-1})$ at this point.

Given a complex manfiold S, a variation of Hodge structure of the given Hodge numbers and polarization on $H_{\mathbb{C}}$ is a holomorphic map

$$\Phi : \, S \to D,$$

such that the tangent map satisfies the Griffiths transversality:

$$\Phi_* : \, T^{1,0}S \to \bigoplus_{k=0}^{n} \mathrm{Hom}(F^k, F^{k-1}).$$

The map Φ is called a period map on S. One denotes the image of any point $p \in S$ by $\Phi(p) = \{F_p^n \subset F_p^{n-1} \subset \cdots \subset F_p^0\}$.

We introduce several notations before we get to the definition of the polarized and marked Calabi-Yau type manifolds. We denote by P_p^k the projection from the horizontal tangent space $\bigoplus_{i=0}^n \mathrm{Hom}(F_p^i, F_p^{i-1})$ of the classifying space to $\mathrm{Hom}(F_p^k, F_p^{k-1})$ at $p \in S$ according to the decomposition:

$$P_p^k : \, \bigoplus_{i=0}^{n} \mathrm{Hom}(F_p^i, F_p^{i-1}) \to \mathrm{Hom}(F_p^k, F_p^{k-1}),$$

and denote the projection from $\bigoplus_{i=0}^n H_p^{i,n-i}$ to $H_p^{n-k,k}$ at $p \in S$ by $P_p^{n-k,k}$ according to the given Hodge decomposition:

$$P_p^{k,n-k} : \, \bigoplus_{i=0}^{n} H_p^{i,n-i} \to H_p^{k,n-k}$$

for each $0 \le k \le n-1$.

2.2. **Calabi-Yau type manifolds.** We generalize the definition of Calabi-Yau type manifolds in [9] as follows,

Definition 2.1. *Let M be a simply connected compact complex projective manifold of $\dim_{\mathbb{C}} M = n$ and L be an ample line bundle over M with $c_1(L)$ be the Kähler class on M. We call M a manifold of Calabi-Yau type if*

(1) The middle dimensional Hodge numbers are similar to that of a Calabi-Yau manifold, that is, there exists some $[n/2] < s \le n$, such that

$$h^{s,n-s}(M) = 1, \quad \text{and} \quad h^{s',n-s'}(M) = 0 \;\; \text{for } s' > s.$$

(2) For any generator $[\Omega] \in H^{s,n-s}(M)$, the contraction map

$$\lrcorner : \, H^{0,1}(M, T^{1,0}M) \to H^{s-1,n-s+1}(M)$$
$$[\varphi] \mapsto [\varphi \lrcorner \Omega]$$

is an isomorphism, where Ω is a $\overline{\partial}$-closed $(s, n-s)$-form.

(3) $H^{s,n-s}(M) = H^{s,n-s}_{pr}(M)$, *and* $H^{2,0}(M)$=*0.*

This paper mainly considers the polarized and marked Calabi-Yau type manifold, which consists of a triple $(M, L, \{\gamma_1, \cdots, \gamma_m\})$ with a Calabi-Yau type manifold M of $\dim_{\mathbb{C}} M = n$, an ample line bundle L over M and a basis $\{\gamma_1, \cdots, \gamma_m\}$ of the integral cohomology group modulo torsion $H^n(M, \mathbb{Z})/Tor$. In the rest of the paper we will use the Chern form $\omega \in c_1(L)$ of the line bundle L as the Kähler form on M. It is well known that each triple $(M, L, \{\gamma_1, \cdots, \gamma_m\})$ gives a polarized Hodge structure on the primitive cohomology group $H^n_{pr}(M, \mathbb{C})$. For details of polarized Hodge structure on primitive cohomology group of Kähler manifolds, we refer the reader to Chapter 7.1 in [23].

Notice that if a manifold M is of Calabi-Yau type, then $H^{s-1,n-s+1}_{pr}(M) = H^{s-1,n-s+1}(M)$. To show this, it is enough to show that the image of the contraction map is in $H^{s-1,n-s+1}_{pr}(M)$, which is proved in the following proposition. This is because condition (2) in Definition 2.1 gives that the contraction map

$$\lrcorner : H^{0,1}(M, T^{1,0}M) \to H^{s-1,n-s+1}(M), \quad [\varphi] \mapsto [\varphi \lrcorner \Omega]$$

is an isomorphism.

For this reason, we simply use $H^n(M, \mathbb{C})$ and $H^{k,n-k}(M)$ to denote the primitive cohomology groups $H^n_{pr}(M, \mathbb{C})$ and $H^{k,n-k}_{pr}(M)$ respectively. And the Hodge numbers $h^{k,n-k} = \dim H^{k,n-k}_{pr}(M)$ are the dimensions of the primitive cohomology groups. By using these notations, we don't change the condition (1) and condition (2) in Definition 2.1.

Proposition 2.2. *Let M be a Calabi-Yau type manifold. Let the contraction map and $[\Omega] \in H^{s,n-s}_{pr}(M)$ be as in Definition 2.1, then for any $[\varphi] \in H^{0,1}(M, T^{1,0}M)$, $[\varphi \lrcorner \Omega] \in H^{s-1,n-s+1}_{pr}(M)$.*

Proof. It is sufficient to show that there exists a smooth $(s, n-s+1)$ form γ, such that $\omega \wedge (\varphi \lrcorner \Omega) = \overline{\partial}\gamma$. Since $[\Omega] \in H^{s,n-s}_{pr}(M)$, there exists a smooth $(s+1, n-s)$ form β, such that $\omega \wedge \Omega = \overline{\partial}\beta$. First we shall show the following two identities:

(i). $\overline{\partial}(\varphi \lrcorner \beta) = \varphi \lrcorner \overline{\partial}\beta$;

(ii). $\varphi \lrcorner (\omega \wedge \Omega) = (\varphi \lrcorner \omega) \wedge \Omega + \omega \wedge (\varphi \lrcorner \Omega)$.

To show (i), denote $\varphi = \sum_i \varphi^i \partial_i$, with $\varphi^i = \sum_j \varphi^i_{\bar{j}} d\bar{z}_j$; $\beta = \sum_I \beta^I dz_I$, and $\beta^I = \sum_J \beta^I_{\bar{J}} d\bar{z}_J$. Then

$$\begin{aligned}
\overline{\partial}(\varphi \lrcorner \beta) &= \overline{\partial}(\sum_i \sum_I \varphi^i \wedge \beta^I \wedge ((-1)^{|J|} \partial_i \lrcorner dz_I)) \\
&= \sum_i \sum_I \overline{\partial}\varphi^i \wedge \beta^I \wedge ((-1)^{|J|} \partial_i \lrcorner dz_I) + \sum_i \sum_I \varphi^i \wedge \overline{\partial}\beta^I \wedge ((-1)^{|J|+1} \partial_i \lrcorner dz_I) \\
&= \sum_i \sum_I \varphi^i \wedge \overline{\partial}\beta^I \wedge ((-1)^{|J|+1} \partial_i \lrcorner dz_I).
\end{aligned}$$

The last identity is due to the property that $[\varphi] \in H^{0,1}(M, T^{1,0}M)$, which gives $\overline{\partial}\varphi = 0$. Then we have the identity (i),

$$\varphi \lrcorner \overline{\partial}\beta = \varphi \lrcorner (\sum_I \overline{\partial}\beta^I dz_I) = \sum_i \sum_I \varphi^i \wedge \overline{\partial}\beta \wedge ((-1)^{|J|+1} \partial_i \lrcorner dz_I) = \overline{\partial}(\varphi \lrcorner \beta).$$

For (ii), we denote $\omega = \sum_k \omega^k dz_k$, with $\omega^k = \sum_j \omega^k_{\bar{j}} d\bar{z}_j$; $\Omega = \sum_I \Omega^I dz_I$, and $\Omega^I = \sum_J \Omega^I_{\bar{J}} d\bar{z}_J$. Then we have

$$\begin{aligned}
&\varphi\lrcorner(\omega \wedge \Omega) = \varphi\lrcorner(\sum_k \sum_I \omega^k dz_k \wedge \Omega^I dz_I) \\
&= -\sum_i \sum_k \sum_I (\varphi^i \wedge \omega^k) \wedge (\partial_i \lrcorner dz_k) \wedge \Omega^I dz_I + (-1)^{|J|} \sum_i \sum_k \sum_I \omega^k dz_k \wedge (\varphi^i \wedge \Omega^I) \wedge (\partial_i \lrcorner dz_I) \\
&= \left(-\sum_i \sum_k (\varphi^i \wedge \omega^k)(\partial_i \lrcorner dz_k) \right) \wedge \left(\sum_I \Omega^I dz_I \right) \\
&\quad + \left(\sum_k \omega^k dz_k \right) \wedge \left((-1)^{|J|} \sum_i \sum_I (\varphi^i \wedge \Omega^I) \wedge (\partial_i \lrcorner dz_I) \right) \\
&= (\varphi\lrcorner\omega) \wedge \Omega + \omega \wedge (\varphi\lrcorner\Omega).
\end{aligned}$$

Moreover, since $[\varphi\lrcorner\omega] \in H^{0,2}(M) \cong H^{2,0}(M) = 0$, there exists $\alpha \in H^{0,1}(M)$, such that $\varphi\lrcorner\omega = \overline{\partial}\alpha$. The fact that Ω is a holomorphic $(s, n-s)$-form implies that $\overline{\partial}\Omega = 0$. Combining all the above properties together, we get

$$\begin{aligned}
\overline{\partial}(\varphi\lrcorner\beta) &= \varphi\lrcorner\overline{\partial}\beta = \varphi\lrcorner(\omega \wedge \Omega) = (\varphi\lrcorner\omega) \wedge \Omega + \omega \wedge (\varphi\lrcorner\Omega) \\
&= \overline{\partial}\alpha \wedge \Omega + \omega \wedge (\varphi\lrcorner\Omega) = \overline{\partial}\alpha \wedge \Omega + \omega \wedge (\varphi\lrcorner\Omega) \\
&= \overline{\partial}(\alpha \wedge \Omega) + \omega \wedge (\varphi\lrcorner\Omega).
\end{aligned}$$

Therefore if we set $\gamma = \varphi\lrcorner\beta - \alpha \wedge \Omega$, we get $\omega \wedge (\varphi\lrcorner\Omega) = \overline{\partial}\gamma$. □

From the definition, one can see that a simply connected Calabi-Yau manifold is of course of Calabi-Yau type. Therefore the results presented in this paper can be applied to simply connected Calabi-Yau manifolds. In the rest of paper, one may consider simply connected Calabi-Yau manifolds as our special cases. We remark that one may refer to [9] for many interesting examples of Calabi-Yau type manifolds of Fano type.

2.3. **Moduli space and Teichmüller space.** The *moduli space* $\mathcal{M}$ of polarized complex structures on a given differential manifold X is a complex analytic space consisting of biholomorphically equivalent pairs of complex structures and ample line bundles (M, L). Let us denote by $[M, L]$ the point in $\mathcal{M}$ corresponding to a pair (M, L), in which M is a complex manifold diffeomorphic to X and L is an ample line bundle on M. If there is a biholomorphic map f between M and M' with $f^*L' = L$, then $[M, L] = [M', L'] \in \mathcal{M}$.

We can similarly define the *Teichmüller space* $\mathcal{T}$ of polarized complex structure by replacing the pair (M, L) by a triple $(M, L, \{\gamma_1, \cdots, \gamma_m)\}$, where $\{\gamma_1, \cdots, \gamma_m)\}$ is a basis of $H^n(M, \mathbb{Z})/\mathrm{Tor}$. For two triples $(M, L, \{\gamma_1, \cdots, \gamma_m\})$ and $(M', L', \{\gamma'_1, \cdots, \gamma'_m)\})$, if there exists a biholomorphic map $f : M \to M'$ with

$$\begin{aligned}
&f^*L' = L, \\
&f^*\gamma'_i = \gamma_i \ \text{ for } \ 1 \le i \le m,
\end{aligned}$$

then $[M, L, \{\gamma_1, \cdots, \gamma_m)\}] = [M', L', \{\gamma'_1, \cdots, \gamma'_m)\}]$. For simplicity we use $[M, L, \gamma]$ to denote the triple $[M, L, \{\gamma_1, \cdots, \gamma_m)\}]$.

In this paper we further assume that the Teichmüller space $\mathcal{T}$ of polarized marked Calabi-Yau type manifolds is a smooth connected complex manifold, and there exists a versal family

$$\pi : \mathcal{U} \to \mathcal{T}$$

containing the polarized and marked Calabi-Yau type manifold $(M, L, \{\gamma_1, \cdots, \gamma_m\})$ as a fiber. Recall that a family of compact complex manifolds $\pi :\ \mathcal{U} \to \mathcal{T}$ is versal at a point $p \in \mathcal{T}$ if it satisfies the following conditions:

(1) If a complex analytic family $\iota :\ \mathcal{V} \to \mathcal{S}$ of compact complex manifolds, a point $s \in \mathcal{S}$, and a bi-holomorphic map $f_0 :\ V = \iota^{-1}(s) \to U = \pi^{-1}(p)$ are given, then there exists a holomorphic map g from a neighbourhood $\mathcal{N}$ of the point s to $\mathcal{T}$ and a holomorphic map $f :\ \iota^{-1}(\mathcal{N}) \to \mathcal{U}$ such that they satisfy the following conditions:

$$g(s) = p,$$
$$f|_{\iota^{-1}(s)} = f_0,$$

with the following commutative diagram

$$\begin{tikzcd} \iota^{-1}(\mathcal{N}) \arrow[r, "f"] \arrow[d, "\iota"] & \mathcal{U} \arrow[d, "\pi"] \\ \mathcal{N} \arrow[r, "g"] & \mathcal{T}. \end{tikzcd}$$

(2) For all g satisfying the above condition, the tangent map $(dg)_s$ is uniquely determined.

If a family $\pi :\ \mathcal{U} \to \mathcal{T}$ is versal at every point $p \in \mathcal{T}$, then it is a versal family on $\mathcal{T}$. In this case the family $\iota^{-1}(\mathcal{N}) \to \mathcal{N}$ is complex analytically isomorphic to the pull back of the family $\pi :\ \mathcal{U} \to \mathcal{T}$ by the holomorphic map g.

The versal family $\pi :\ \mathcal{U} \to \mathcal{T}$ is local Kuranishi at any point $p \in \mathcal{T}$, and this implies that the Kodaira-Spencer map

$$\kappa :\ T^{1,0}_p\mathcal{T} \to H^{0,1}(M_p, T^{1,0}M_p),$$

for each $p \in \mathcal{T}$ is an isomorphism. We refer the reader to page 8-10 in [18], page 94 in [14] or page 19 in [22], for more details about versal families and local Kuranishi families, and Chapter 4 in [12] for more details about deformation of complex structures and the Kodaira-Spencer map.

For a polarized Calabi-Yau type manifold (M, L), let

$$\mathrm{Aut}(M, L) = \{\alpha :\ M \to M|\ \alpha^*L = L\}$$

be the group of biholomorphic maps on M preserving the polarization L. We have a natural representation of $\mathrm{Aut}(M, L)$,

$$\sigma :\ \mathrm{Aut}(M, L) \to \mathrm{Aut}(H^n(M, \mathbb{Z}))$$
$$\alpha \mapsto \alpha^*.$$

The following theorem tells us that the Techmüller space of polarized and marked Calabi-Yau type manifolds described as above is simply connected.

Theorem 2.3. *The Techmüller space $\mathcal{T}$ of polarized and marked Calabi-Yau type manifolds is simply connected.*

Proof. Suppose $\mathcal{T}$ is not simply connected, and $\widetilde{\mathcal{T}}$ is the universal cover of $\mathcal{T}$ with covering map

$$\pi_{\widetilde{\mathcal{T}}} : \widetilde{\mathcal{T}} \to \mathcal{T}.$$

For each point $p = [M, L, \gamma]$ in $\mathcal{T}$ the pre-image is $\pi_{\widetilde{\mathcal{T}}}^{-1}(p) = \{p_i | \ i \in I\} \subset \widetilde{\mathcal{T}}$ and $|I| > 1$. Let $\widetilde{\pi} : \ \widetilde{\mathcal{U}} \to \widetilde{\mathcal{T}}$ be the pull back family of $\pi : \ \mathcal{U} \to \mathcal{T}$ with the following commutative diagram,

$$\begin{tikzcd}
\widetilde{\mathcal{U}} \arrow[r] \arrow[d, "\widetilde{\pi}"] & \mathcal{U} \arrow[d, "\pi"] \\
\widetilde{\mathcal{T}} \arrow[r, "\pi_{\widetilde{\mathcal{T}}}"] & \mathcal{T}.
\end{tikzcd}$$

Then $\widetilde{\pi} : \ \widetilde{\mathcal{U}} \to \widetilde{\mathcal{T}}$ is also a versal family of polarized and marked Calabi-Yau type manifolds.

Then for each different i and $j \in I$, there exists an element α in the deck transformation group of the covering map, such that $\alpha(p_i) = p_j$. Because $\pi_{\widetilde{\mathcal{T}}}(p_i) = \pi_{\widetilde{\mathcal{T}}}(p_j) = [M, L, \gamma]$, this α can be viewed as a biholomorphic map on M which preserves the marking γ. That means $\alpha \in \ker(\sigma)$. In the following lemma, we will show that we can extend the action of $\ker \sigma$ to $\widetilde{\mathcal{U}}$, which leaves $\widetilde{\mathcal{T}}$ fixed and also fixing the polarization on each fiber of the family, thus induces the action of $\ker \sigma$ on $\widetilde{\mathcal{T}}$. Furthermore, we will show in the following corollary that the action of $\ker \sigma$ on $\widetilde{\mathcal{T}}$ is a trivial action.

Lemma 2.4. *Let (M, L, γ) be a fiber of the versal family $\widetilde{\pi} : \ \widetilde{\mathcal{U}} \to \widetilde{\mathcal{T}}$. Then for any $\alpha \in \ker \sigma$, there is an extension $\widetilde{\alpha}$ on $\widetilde{\mathcal{U}}$ leaving the base space $\widetilde{\mathcal{T}}$ fixed and also fixing the polarization on each fiber of the family.*

Proof. Let $p \in \widetilde{\mathcal{T}}$ be a point in the Teichmüller space with $\widetilde{\pi}^{-1}(p) = (M, L, \gamma)$. The local universality of the family $\widetilde{\pi} : \ \widetilde{\mathcal{U}} \to \widetilde{\mathcal{T}}$ shows that there exists a neighbourhood U_p of $p \in \widetilde{\mathcal{T}}$ with holomorphic morphisms $\widetilde{\alpha} : \widetilde{\pi}^{-1}(U_p) \to \widetilde{\mathcal{U}}$ and $f : \ U_p \to \widetilde{\mathcal{T}}$, such that the following diagram is commutative,

$$\begin{tikzcd}
\widetilde{\pi}^{-1}(U_p) \arrow[r, "\widetilde{\alpha}"] \arrow[d, "\widetilde{\pi}"] & \widetilde{\mathcal{U}} \arrow[d, "\widetilde{\pi}"] \\
U_p \arrow[r, "f"] & \widetilde{\mathcal{T}}.
\end{tikzcd}$$

To show $\widetilde{\alpha}$ leaves the base space $\widetilde{\mathcal{T}}$ fixed, it is sufficient to show that f is the identity on U_p. Indeed $f(p) = p$ from the definition of f, suppose f is not an identity on U_p. Then the tangent map $f_* : \ T_pU_p \to T_p\widetilde{\mathcal{T}}$ is not identity either. By the identification of

$T_pU_p = T_p\widetilde{\mathcal{T}}$ with $H^{0.1}(M, T^{1,0}M)$ we have the following commutative diagram,

$$\begin{array}{ccc}
T_p^{1,0}\widetilde{\mathcal{T}} & \xrightarrow{f_*^{-1}} & T_p^{1,0}U_p \\
\downarrow & & \downarrow \\
H^{0.1}(M, T^{1,0}M) & & H^{0.1}(M, T^{1,0}M) \\
\downarrow & & \downarrow \\
H^{s-1,n-s+1}(M) & \xrightarrow{\alpha^*} & H^{s-1,n-s+1}(M).
\end{array}$$

But $\alpha \in \ker(\sigma)$ implies the map $\alpha^* : \ H^{s-1,n-s+1}(M) \to H^{s-1,n-s+1}(M)$ is an identity. This contradicts the assumption that f_* is not identity.

Next we show that for each $q \in U_p$ the biholomorphic map $\widetilde{\alpha}_q$ on the fiber M_q preserves the polarization L. Because $H^2(M, \mathbb{Z})$ is a discrete group, we have that, for any point $q \in U_p$,

$$c_1(\widetilde{\alpha}_q^* L) = c_1(\widetilde{\alpha}_p^* L) = c_1(L).$$

The simply connected property of M implies holomorphic line bundles on M are uniquely determined by the first Chern class. Therefore for any $q \in U_p$, we have $\widetilde{\alpha}_q^*(L) = L$.

Now let us define a sheaf $\Im$ on the base space $\widetilde{\mathcal{T}}$ as follows, for any open set $U \subset \widetilde{\mathcal{T}}$, we assign the group $\Im(U)$ to be all the biholomorphic maps $\alpha_{_U} : \ \widetilde{\pi}^{-1}(U) \to \widetilde{\pi}^{-1}(U)$ which leaves the open set U fixed and preserving the polarization on each fiber. In another word for any $\alpha_{_U} \in \Im(U)$, we have the following commutative diagram,

$$\begin{array}{ccc}
\widetilde{\pi}^{-1}(U) & \xrightarrow{\alpha_U} & \widetilde{\pi}^{-1}(U) \\
\downarrow{\scriptstyle \widetilde{\pi}} & & \downarrow{\scriptstyle \widetilde{\pi}} \\
U & \xrightarrow{id} & U.
\end{array}$$

and for each point $q \in U$, the restriction of $\alpha_{_U}$ on the fiber $M_q = \widetilde{\pi}^{-1}(q)$ preserves the polarization L over M_q. If $V \subset U$ is open then the restriction map of the sheaf is given by,

$$\begin{aligned} res : \ & \Im(U) \to \Im(V) \\ & \alpha_U \to \alpha_U|_{\widetilde{\pi}^{-1}(V)}. \end{aligned}$$

From the local extension result discussed as above, we have that for any point $p \in \widetilde{\mathcal{T}}$, there exists a neighborhood $U_p \subset \widetilde{\mathcal{T}}$ such that any $\alpha \in \ker\sigma$ on M_p can be extended to the family $\widetilde{\pi}^{-1}(U_p)$. This means the restriction map

$$\begin{aligned} res : \ & \Im(U_p) \to \ker\sigma \\ & \alpha_{_U} \mapsto \alpha_{_U}|_{M_p} \end{aligned}$$

is an isomorphism. Therefore the sheaf $\Im$ is a locally constant sheaf. Using the fact that $\widetilde{\mathcal{T}}$ is simply connected and Proposition 3.9 in [24], we have $\Im$ is a constant sheaf. This means $\Im(\widetilde{\mathcal{T}}) = \ker\sigma$, so for each point $p \in \widetilde{\mathcal{T}}$ and $\alpha \in \ker\sigma$, there is a global section

$\widetilde{\alpha} \in \Im(\widetilde{\mathcal{T}})$ such that $\widetilde{\alpha}|_{M_p} = \alpha$. By the definition of the sheaf $\Im$, we have the commutative diagram,

$$\begin{tikzcd} \widetilde{\mathcal{U}} \arrow[r, "\widetilde{\alpha}"] \arrow[d, "\widetilde{\pi}"] & \widetilde{\mathcal{U}} \arrow[d, "\widetilde{\pi}"] \\ \widetilde{\mathcal{T}} \arrow[r, "id"] & \widetilde{\mathcal{T}}. \end{tikzcd}$$

And restricted to each fiber M_q the morphism $\widetilde{\alpha}|_{M_q}$ preserves the polarization L. $\square$

We remark that we are using the argument in Lemma 2.6 in [19] for the first part of proof of Lemma 2.4. As a corollary of above lemma we have the following result, which will be useful in the proof of Theorem 2.3.

Corollary 2.5. *The action of ker(σ) on $\widetilde{\mathcal{T}}$ is trivial.*

Proof. For each element $\alpha \in \ker(\sigma)$, we have a global extension $\widetilde{\alpha}$ acts on the family $\widetilde{\mathcal{U}}$ with the commutative diagram

$$\begin{tikzcd} \widetilde{\mathcal{U}} \arrow[r, "\widetilde{\alpha}"] \arrow[d, "\pi"] & \widetilde{\mathcal{U}} \arrow[d, "\pi"] \\ \widetilde{\mathcal{T}} \arrow[r, "\overline{f}"] & \widetilde{\mathcal{T}}. \end{tikzcd}$$

Then α acts on $\widetilde{\mathcal{T}}$ by the holomorphic map $\overline{f}: \widetilde{\mathcal{T}} \to \widetilde{\mathcal{T}}$. But Lemma 2.4 implies that $\overline{f}$ is an identity map. Therefore the action of α on $\widetilde{\mathcal{T}}$ is trivial. $\square$

But the deck transformation $\alpha: \mathcal{T} \to \mathcal{T}$ can also be viewed as a biholomorphic map

$$\overline{\alpha}: \widetilde{\mathcal{U}} \to \widetilde{\mathcal{U}}.$$

This is the extension of the bi-holomorphic map $\alpha : M \to M$, but the Corollary 2.5 shows that the global extension $\widetilde{\alpha} : \widetilde{\mathcal{U}} \to \overline{\mathcal{U}}$ should leave the $\widetilde{\mathcal{T}}$ fixed, this contradicts the fact that $\alpha(p_i) = p_j \neq p_i$.

$\square$

2.4. **The period map on the Teichmüller space.** As mentioned in Section 2.2, in the rest of the paper, we simply use $H^n(M,\mathbb{C})$ and $H^{k,n-k}(M)$ to denote the primitive cohomology groups $H^n_{pr}(M,\mathbb{C})$ and $H^{k,n-k}_{pr}(M)$ respectively. And the Hodge numbers $h^{k,n-k} = \dim H^{k,n-k}_{pr}(M)$ are the dimensions of the primitive cohomology groups. By using these notations, we don't change the condition (1) and condition (2) in Definition 2.1.

For any point $p \in \mathcal{T}$, let M_p be the fiber of the family $\pi: \mathcal{U} \to \mathcal{T}$ at p, which is a polarized and marked Calabi-Yau type manifold. Since the Teichmüller space is simply connected and we have fixed the basis of the middle cohomology group modulo torsions, we can use this to identify $H^n(M_p,\mathbb{C})$ for all fibers on $\mathcal{T}$, and thus get a canonical trivial bundle $H^n(M_p,\mathbb{C}) \times \mathcal{T}$. We have similar identifications for $H^n(M_p,\mathbb{Q})$ and $H^n(M_p,\mathbb{Z})/Tor$. The Hodge numbers $\{h^{s,n-s}, h^{s-1,n-s+1}, \cdots, h^{n-s,s}\}$ of M_p are indepentdent to the choice of $p \in \mathcal{T}$. Therefore we have corresponding data of $H^n(M,\mathbb{C}), H^n(M,\mathbb{Z}), \{h^{k,n-k}\}$ and the intersection form on $H^n(M,\mathbb{C})$ on a given polarized and marked Calabi-Yau type manifolds.

Let D be the period domain defined as in Section 2.1. The period map

$$\Phi : \mathcal{T} \to D$$

from the Teichmüller space of polarized and marked Calabi-Yau type manifolds to the period domain is defined by assigning each point $p \in \mathcal{T}$ the Hodge structure on M_p.

We denote $\Phi(p) = \{F^s(M_p) \subset \cdots \subset F^{n-s}(M_p)\}$, $F^k(M_p)$ by F_p^k and $H^{k,n-k}(M_p)$ by $H_p^{k,n-k}$ for convenience.

Recall that we have described general period maps in Section 2.1. Here we remark that period maps have several nice properties, the reader may refer to Chapter 10 in [23] for details. Among these properties, the one we are most interested in is the following Griffiths transversality on the period map $\Phi : \mathcal{T} \to D$ from the Teichmüller space of polarized and marked Calabi-Yau type manifolds to the period domain, that is, for any $p \in \mathcal{T}$ and $v \in T_p^{1,0}\mathcal{T}$, the tangent map satisfies,

$$\Phi_*(v) \in \bigoplus_{k=n-s}^{s} \mathrm{Hom}\left(F_p^k/F_p^{k+1}, F_p^{k-1}/F_p^k\right), \tag{7}$$

where $F^{s+1} = 0$, or equivalently

$$\Phi_*(v) \in \bigoplus_{k=n-s}^{s} \mathrm{Hom}(F_p^k, F_p^{k-1}),$$

such that for any $v \in T_p^{1,0}\mathcal{T}$ and $[\alpha] \in F_p^k$,

$$\Phi_*(v)([\alpha]) = [\kappa(v) \lrcorner \alpha].$$

For the period map on the Teichmüller space of polarized and marked Calabi-Yau type manifold, we have the following property.

Proposition 2.6. *For any $p \in \mathcal{T}$ and any generator $[\Omega_p]$ of F_p^s, the tangent map Φ_*, the map*

$$P_p^s \circ \Phi_* : \quad T_p^{1,0}\mathcal{T} \cong H^{0,1}(M_p, T^{1,0}M_p) \to Hom(F_p^s, F_p^{s-1}/F_p^s) \cong H_p^{s-1,n-s+1}$$

is an isomorphism.

Proof. The first isomorphism

$$T_p^{1,0}\mathcal{T} \cong H^{0,1}(M_p, T^{1,0}M_p)$$

follows from the condition for Kuranishi families that the Kodair-Spencer map is an isomorphism. The second isomorphism

$$\mathrm{Hom}(F_p^s, F_p^{s-1}/F_p^s) \cong H_p^{s-1,n-s+1}$$

follows from the condition on Calabi-Yau type manifold that $\dim F_p^s = 1$. This isomorphism is determined by the choice of the generator $[\Omega_p]$.

Now it is clear that the map

$$P_p^s \circ \Phi_* : \; H^{0,1}(M_p, T^{1,0}M_p) \to H_p^{s-1,n-s+1}$$

is given by contraction

$$P_p^s \circ \Phi_*(v) = [\kappa(v) \lrcorner \Omega_p].$$

This contraction map is an isomorphism by second condition in the definition of Calabi-Yau type manifolds. The proof is complete. □

To end this section, we introduce corresponding notations for the Hodge bundles for Calabi-Yau type manifolds. Hodge bundles over $\mathcal{T}$ are the pull-backs of the Hodge bundles over $\check{D}$ through the period map. For convenience, we still denote them by F^k, for each $n-s \leq k \leq s$. We will also denote by P^k_p the projection from $H^n(M_p,\mathbb{C})$ to F^k_p with respect to the Hodge filtration on M_p, and $P^{n-k,k}_p$ the projection from $H^n(M_p,\mathbb{C})$ to $H^{n-k,k}_p$ according to the Hodge decomposition on M_p.

3. Holomorphic affine structure on the Teichmüller space

In this setion we will construct a holomorphic affine structure on the Teichmüller space of polarized and marked Calabi-Yau type manifolds. In Section 3.1, we will give a brief review on definitions and some basic properties of affine manifolds and affine maps. One may refer to page 215 of [13], page 231 of [25], page 141 of [2] for more details. In Section 3.2 we construct a local coordinate system around each point of the Teichmüller space $\mathcal{T}$ of polarized and marked Calabi-Yau type manifolds, which gives the Kuranishi coordinate cover on $\mathcal{T}$. In Section 3.3, we prove that this Kuranishi coordinate cover is a holomorphic affine cover, thus it gives the holomorphic affine structure on the Teichmüller space.

In this section, we will freely use the notations of the Hodge numbers, Hodge decompositions and Hodge filtrations from the definition of Calabi-Yau type manifolds in the previous section.

3.1. Affine manifolds and affine maps.

Definition 3.1. *Let M be a differentiable manifold of real dimension n, if there is a coordinate cover $\{(U_i,\, \varphi_i);\, i \in I\}$ of M satisfying that $\varphi_{ik} = \varphi_i \circ \varphi_k^{-1}$ is a real affine transformation on $\mathbb{R}^n$, whenever $U_i \cap U_k$ is not empty, then we call that $\{(U_i,\, \varphi_i);\, i \in I\}$ is a real affine coordinate cover on M and it defines a real affine structure on M.*

Similarly one can define the concepts of holomorphic affine coordinate cover and holomorphic affine structure on a complex manifold M:

Definition 3.2. *Let M be a complex manifold of complex dimension n, if there is a coordinate cover $\{(U_i,\, \varphi_i);\, i \in I\}$ of M satisfying that $\varphi_{ik} = \varphi_i \circ \varphi_k^{-1}$ is a holomorphic affine transformation on $\mathbb{C}^n$, whenever $U_i \cap U_k$ is not empty, then we call that $\{(U_i,\, \varphi_i);\, i \in I\}$ is a holomorphic affine coordinate cover on M and it defines a holomorphic affine structure on M.*

We have the following concept of the holomorphic flat connection, which is one-to-one correspondent to the holomorphic affine structures on a complex manifold. For details about this, we refer the reader to page 216 of [13], page 233-234 of [25] or page 140 of [2]. The proof for the one-to-one correspondence between holomorphic flat connections and holomorphic affine structures can be found in page 217-219 of [13] or Section 3 of [25].

Let M be a complex manifold, and $\Gamma(M,TM)$ denote the space of smooth sections of the tangent bundle of M. A linear connection ∇ on M is defined to be a bilinear map

$$\Gamma(M,TM)\times\Gamma(M,TM)\to\Gamma(M,TM), \qquad (X,Y)\mapsto\nabla_X Y$$

satisfying the following conditions,

(i) $\nabla_{fY}X = f\nabla_Y X$;
(ii) $\nabla_Y fX = f(\nabla_Y X) + Y(f)X$

for any $f \in C^\infty(M)$.

A linear connection ∇ is called a holomorphic linear connection if the following two more conditions are satisfied,

(iii) $(\nabla_Y X)^{1,0} = \nabla_Y X^{1,0}$;
(iv) If V and W are complex holomorphic vector fields defined on an open subset $U \subset M$, then $\nabla_W V$ is also holomorphic on U.

A holomorphic linear connection ∇ is called a holomorphic flat connection if the following two additional conditions are satisfied,

(v) $\nabla_X Y - \nabla_Y X = [X, Y]$;
(vi) $\nabla_X \nabla_Y - \nabla_Y \nabla_X - \nabla_{[X,Y]} = 0$.

Condition (v) and (vi) is saying that a holomorphic flat connection ∇ is a holomorphic linear connection which is torsion-free and has zero curvature.

Now let us consider affine maps between holomorphic affine manifolds.

Let M and M' be two holomorphic affine manifolds with holomorphic coordinate covers $\{U_i, \varphi_i\}$ and $\{V_j, \psi_j\}$ respectively. They both have natural holomorphic flat connections according to the one-to-one correspondence between holomorphic flat connections and holomorphic affine structures. Let $f : M \to M'$ be a smooth map. We call $f : M \to M'$ a holomorphic affine map if the induced map $f_* : TM \to TM'$ maps every horizontal curve into a horizontal curve, that is, f maps each parallel vector field along each curve $\gamma(t)$ of M into a parallel vector field along the curve $f(\gamma(t))$. Especially if the tangent vector field $\frac{d}{dt}\gamma(t)$ is parallel along $\gamma(t)$, then the image of this vector field is also parallel along $f(\gamma(t))$, that is, f maps geodesics on M to geodesics on M'. Since the geodesics on an affine manifold are straight lines in the holomorphic affine coordinate charts, we deduce that a smooth map $f : M \to M'$ is a holomorphic affine map if and only for any $f^{-1}(V_j) \cap U_i \neq \emptyset$, the following map

$$\psi_j \circ f \circ \varphi_i^{-1} : \ \varphi_i(f^{-1}(V_j) \cap U_i) \to \psi_j(V_j) \tag{8}$$

is a holomorphic affine transformation. Here $\varphi_i(f^{-1}(V_j) \cap U_i)$ and $\psi_j(V_j)$ are both open sets in complex Euclidean spaces.

In particular, for later use, we mention the following theorem about the extension of holomorphic affine maps, which is the holomorphic analogue of Theorem 6.1 in [11]. We refer the reader to page 252-255 of [11] and Section 4 of [8] for the proof.

Theorem 3.3. *Let M be a connected, simply connected holomorphic affine manifold and M' be a complete holomorphic affine manifold. Then any holomorphic affine map f_U on a connected open subset U of M into M' can be uniquely extended to a holomorphic affine map f on M into M'.*

3.2. **Local holomorphic coordinates on the Teichmüller space.** For any fixed point $p \in \mathcal{T}$, we choose orthonormal bases $\{\eta_0\}$ and $\{\eta_1, \cdots, \eta_N\}$ of $H_p^{s,n-s}$ and $H_p^{s-1,n-s+1}$ respectively. Let U_p be a small neighbourhood around p, and $[\Omega_p]$ be a local holomorphic section of the Hodge bundle $F^s = H^{s,n-s}$, i.e.

$$[\Omega_p] : U_p \to F^s = H^{s,n-s},$$

such that $[\Omega_p](q) \in F_q^s = H_q^{s,n-s}$. Without loss of generality we also assume $[\Omega_p](p) = \eta_0$. Since as complex vector spaces, $H^n(M_p, \mathbb{C}) = H^n(M_q, \mathbb{C})$ for any $q \in U_p$, we have

$$P_p^{s,n-s}([\Omega_p](q)) = a_0(q)\eta_0, \tag{9}$$

$$P_p^{s-1,n-s+1}([\Omega_p](q)) = a_1(q)\eta_1 + \cdots + a_N(q)\eta_N, \tag{10}$$

for certain holomorphic functions $a_i(q)$ in q. This gives us a collection of local holomorphic functions on U_p,

$$\tau_i : U_p \to \mathbb{C}, \quad \tau_i(q) = a_i(q)/a_0(q), \quad \text{for } 1 \le i \le N. \tag{11}$$

In particular, $\tau_i(p) = 0$ for $1 \le i \le N$.

In the following proposition we will show that these functions actually give a local holomorphic coordinate system on U_p.

Proposition 3.4. *The holomorphic map on U_p,*

$$\rho_{U_p} := (\tau_1, \cdots, \tau_N) : U_p \to \mathbb{C}^N$$

is a local embedding.

Proof. The proof follows directly from Proposition 2.6, which shows that

$$P_p^s \circ \Phi_* : T_p^{1,0}\mathcal{T} \to H_p^{s-1,n-s+1}$$

is an isomorphism. Let $\{t_1, \cdots, t_N\}$ be an arbitrary local holomorphic coordinate system around p, then we have the Jacobian matrix $[\partial a_i/\partial t_j]_{i,j=0}^N$ which is non-singular at p. This means $\{a_1, \cdots, a_N\}$ gives a local holomorphic coordinate system around p.

Since $\tau_i = a_i/a_0$, we have

$$\frac{\partial \tau_i}{\partial t_j} = \frac{\dfrac{\partial a_i}{\partial t_j}a_0 - \dfrac{\partial a_0}{\partial t_j}a_i}{a_0^2}, \quad \text{for } 1 \le i \le N.$$

Since $a_0(p) = 1$, and $a_i(p) = 0$, thus $\dfrac{\partial \tau_i}{\partial t_j}(p) = \dfrac{\partial a_i}{\partial t_j}(p)$. Therefore $\{\tau_1, \cdots, \tau_N\}$ also gives a local holomorphic coordinate system around p. This proves that the map ρ_{U_p} is a local embedding. □

Notice that the definition of these coordinate functions depends on the choices of the basis $\{\eta_0, \eta_1, \cdots, \eta_N\}$ of $H_p^{s,n-s} \oplus H_p^{s-1,n-s+1}$. We call such local holomorphic coordinates $\{\tau_1, \cdots, \tau_N\}$ the local holomorphic coordinates associated to $\{\eta_0, \eta_1, \cdots, \eta_N\}$.

In general, for any point $q \in \mathcal{T}$, by choosing bases of $H_q^{s,n-s}$ and $H_q^{s-1,n-s+1}$, we can define such a local holomorphic coordinate $(U_q, \{\sigma_1, \cdots, \sigma_N\})$ around q in the same way. We call the collection $\{(U_q, \{\sigma_1, \cdots, \sigma_N\}), q \in \mathcal{T}\}$ constructed in this way the Kuranishi coordinate cover over $\mathcal{T}$, and each local holomorphic coordinate system $(U_q, \{\sigma_1, \cdots, \sigma_N\})$ a Kuranishi coordinate chart around q.

For $p \in \mathcal{T}$, by using the Kuranishi coordinate chart $(U_p, \{\tau_1, \cdots, \tau_N\})$, we can express a holomorphic class in $H_q^{s,n-s}$ at each point $q \in U_p$ in the following Taylor expansion form:

$$[\Omega_{p,\tau}^c](q) = [\Omega_p](q)/a_0(q) = \eta_0 + \tau_1(q)\eta_1 + \cdots + \tau_N(q)\eta_N + g(q), \tag{12}$$

with $g(q) \in \bigoplus_{k=2}^{s} H_p^{s-k,n-s+k}$. This gives us a local holomorphic section of F^s on U_p:

$$[\Omega_{p,\tau}^c] : U_p \to F^s, \qquad q \mapsto [\Omega_{p,\tau}^c](q).$$

We call $[\Omega_{p,\tau}^c]$ the local canonical section of F^s around the point p in the local holomorphic coordinate system $(U_p, \{\tau_1, \cdots, \tau_N\})$.

To close this subsection, we remark that different choices of the bases of $H_p^{s,n-s} \oplus H_p^{s-1,n-s+1}$ won't affect the affine structure that we will show on the Teichmüller space $\mathcal{T}$ of polarized and marked Calabi-Yau type manifolds in the following sense.

For a fixed point $p \in \mathcal{T}$, we can construct, as in Section 3.2, the local holomorphic coordinate charts $(U_p, \{\tau_1, \cdots, \tau_N\})$ and $(U_p, \{\tau_1', \cdots, \tau_N'\})$ associated to two different bases $\{\eta_0, \eta_1, \cdots, \eta_N\}$ and $\{\eta_0', \eta_1', \cdots, \eta_N'\}$ of $H_p^{s,n-s} \oplus H_p^{s-1,n-s+1}$ respectively, then we can conclude:

Lemma 3.5. *The transition map between $(U_p, \{\tau_1, \cdots, \tau_N\})$ and $(U_p, \{\tau_1', \cdots, \tau_N'\})$ is a holomorphic affine map.*

Proof. Let A be the $N \times N$ matrix, such that $(\eta_1, \cdots, \eta_N) = (\eta_1', \cdots, \eta_N')A$, and $\eta_0 = c\eta_0'$, for some $c \in \mathbb{C}$. Using the same notations as in (9), (10) and (11), we define

$$\tau_i = a_i/a_0, \qquad \text{and} \quad \tau_i' = a_i'/a_0', \quad \text{for } 1 \leq i \leq N.$$

Then we have the canonical holomorphic classes $[\Omega_{p,\tau}^c](q)$ and $[\Omega_{p,\tau'}^c](q)$ at $q \in U_p$ expressed in different local holormorphic coordinates, respectively:

$$[\Omega_{p,\tau}^c](q) = [\Omega_p](q)/a_0(q) = \eta_0 + \tau_1(q)\eta_1 + \cdots + \tau_N(q)\eta_N + g(q),$$

$$[\Omega_{p,\tau'}^c](q) = [\Omega_p](q)/a_0'(q) = \eta_0' + \tau_1(q)\eta_1' + \cdots + \tau_N(q)'\eta_N + g'(q),$$

with $g(q), g'(q) \in \bigoplus_{k=n-s}^{s-2} H_p^{k,n-k}$. Thus, we have

$$[\Omega_p](q) = a_0(q)\left(\eta_0 + \sum_{i=1}^{N} \tau_i(q)\eta_i + g(q)\right) = a_0'(q)\left(\eta_0' + \sum_{i=1}^{N} \tau_i'(q)\eta_i' + g'(q)\right).$$

By comparing the coefficients of $(s, n-s)$ type part of $[\Omega_p](q)$, we get,

$$a_0(q)c\eta_0' = a_0(q)\eta_0 = a_0'(q)\eta_0',$$

thus $\dfrac{a_0'(q)}{a_0(q)} = c$.

By comparing the coefficients of the $(s-1, n-s+1)$ type part of $[\Omega_p](q)$, we get,

$$(\eta_1, \cdots, \eta_N)(\tau_1, \cdots, \tau_N)^T = c(\eta_1', \cdots, \eta_N')(\tau_1', \cdots, \tau_N')^T = c(\eta_1, \cdots, \eta_N)A(\tau_1', \cdots, \tau_N')^T.$$

which implies

$$(\tau_1, \cdots, \tau_N)^T = cA(\tau_1', \cdots, \tau_N')^T.$$

Thus the transition map between $(\tau_1, \cdots, \tau_N)^T$ and $(\tau_1', \cdots, \tau_N')^T$ is cA, which is holomorphic affine. This completes the proof of the lemma. $\square$

3.3. **Holomorphic affine structure on the Teichmüller space $\mathcal{T}$.**
We say an orthonormal basis

$$\xi = \left\{\xi_0, \xi_1, \cdots, \xi_N, \cdots, \xi_{f^{k+1}}, \cdots, \xi_{f^k-1}, \cdots, \xi_{f^{n-s+2}}, \cdots, \xi_{f^{n-s+1}-1}, \xi_{f^{n-s}-1}\right\}$$

of $H^n(M,\mathbb{C})$ is an adapted basis for the given Hodge decomposition

$$H^n(M,\mathbb{C}) = H^{s,n-s} \oplus H^{s-1,n-s+1} \oplus \cdots \oplus H^{n-s+1,s-1} \oplus H^{n-s,s},$$

if it satisfies

$$F^k/F^{k+1} = H^{k,n-k} = \mathrm{Span}_{\mathbb{C}}\left\{\xi_{f^{k+1}}, \cdots, \xi_{f^k-1}\right\}.$$

Recall in Section 2.1, we can identify a point $\Phi(p) = \{F_p^s \subset F_p^{s-1} \subset \cdots \subset F_p^{n-s}\} \in D$ with its Hodge decomposition $\bigoplus_{k=n-s}^s H_p^{k,n-k}$. Thus we have the notion of an adapted basis for the Hodge decomposition $\Phi(p)$ for each $p \in \mathcal{T}$. In the rest of the paper, we will write an adapted basis for the Hodge decomposition of $H^n(M_p,\mathbb{C})$ at each point $p \in \mathcal{T}$ in the row vector form,

$$\xi = (\xi_0, \xi_1, \cdots, \xi_N, \cdots, \xi_{f^{k+1}}, \cdots, \xi_{f^k-1}, \cdots, \xi_{f^{n-s+2}}, \cdots, \xi_{f^{n-s+1}-1}, \xi_{f^{n-s}-1}).$$

Recall that a square matrix $T = [T^{\alpha,\beta}]$, with each $T^{\alpha,\beta}$ a submatrix, is called a block lower triangular matrix if $T^{\alpha,\beta}$ is zero matrix whenever $\alpha < \beta$. We use T_{ij} to denote the entries of the matrix T.

Let $\Phi : \mathcal{T} \to D$ be the period map from the Teichmüller space of polarized and marked Calabi-Yau type manifolds to the period domain. For a fixed point $p \in \mathcal{T}$, we construct the local holomorphic coordinate system $(U_p, \{\tau_1, \cdots, \tau_N\})$ around p associated to a given basis $\{\xi_0(p), \xi_1(p), \cdots, \xi_N(p)\}$ of $H_p^{s,n-s} \oplus H_p^{s-1,n-s+1}$ as in Section 3.2. We complete the basis $\{\xi_0(p), \xi_1(p), \cdots, \xi_N(p)\}$ to an adapted basis $\{\xi_0(p), \cdots, \xi_{m-1}(p)\}$ for the Hodge decomposition of $\Phi(p)$. Then we will have the following lemma.

Lemma 3.6. *For an adapted basis $\{\xi_0(\tau), \cdots, \xi_{m-1}(\tau)\}$ for the Hodge decomposition of $\Phi(\tau)$ for any $\tau \in U_p$, there exists a non-singular block lower trangular matrix $T(\tau)$ such that*

$$(\xi_0(\tau), \cdots, \xi_{m-1}(\tau)) = (\xi_0(p), \cdots, \xi_{m-1}(p))T(\tau).$$

Proof. This lemma is a direct corollary of the Griffiths transversality for the period map from the Teichmüller space of polarized and marked Calabi-Yau type manifolds to the period domain. In this proof, we identify the notation for a point $\tau \in U_p$ with its corresponding coordinates in the local coordinate system $(U_p, \{\tau_1, \cdots \tau_N\})$.

Consider a local holomorphic family of adapted bases $\{\xi_0, \cdots, \xi_{m-1}\}$ for the Hodge decompositions of the Hodge structures over U_p, obtained by trivializing the Hodge bundles in a small neighbourhood of p, such that $\{\xi_0(p), \cdots, \xi_{m-1}(p)\}$ is the given adapted basis for the Hodge decomposition of $\Phi(p)$. That is, for any $\tau \in U_p$, $\{\xi_0(\tau), \cdots, \xi_{m-1}(\tau)\}$ gives an adapted Hodge basis for the Hodge decomposition of $\Phi(\tau)$. We need to show that for any $\tau \in U_p$, the adapted basis $\{\xi_0(\tau), \cdots, \xi_{m-1}(\tau)\}$ for the Hodge decomposition of $\Phi(\tau)$ given by this local holomorphic family is differed from $\{\xi_0(p), \cdots, \xi_{m-1}(p)\}$ by a non-singular block lower triangular matrix. Let

$$\{F_\tau^s \subset F_\tau^{s-1} \subset \cdots \subset F_\tau^{n-s}\} = \Phi(\tau)$$

be the Hodge filtration determined for the Hodge decomposition of $\Phi(\tau)$. Then the Griffiths transversality implies that

$$\frac{\partial}{\partial\tau}(F^k_\tau/F^{k+1}_\tau)\subset F^{k-1}_\tau/F^k_\tau, \quad \text{for all } n-s\le k\le s \tag{13}$$

where the quotient bundle F^k/F^{k+1} is a holomorphic vector bundle over $\mathcal{T}$ and has the relation $F^k/F^{k+1}=H^{k,n-k}$.

For each $f^{k+1}\le j\le f^k-1$, we have $\xi_j(\tau)\in F^k_\tau/F^{k+1}_\tau=H^{k,n-k}_\tau$, and by using (13), we get

$$\frac{\partial}{\partial\tau^l}\xi_j(\tau)\in F^{k-1}_\tau/F^k_\tau=H^{k-1,n-k+1}_\tau \ \text{ for each } \ 1\le l\le N.$$

With the convention of multiple derivative notations, we get

$$\frac{\partial^{|I|}}{\partial\tau^I}\xi_j(\tau)\in F^{k-|I|}_\tau/F^{k+1-|I|}_\tau=H^{k-|I|,n-k+|I|}_\tau \ \text{ for each } \ |I|\ge 0.$$

Thus we can write $\frac{\partial^{|I|}}{\partial\tau^I}\xi_j(\tau)$ as a linear combination of $\{\xi_{f^{k+1-|I|}}(\tau),\cdots,\xi_{f^{k-|I|}-1}(\tau)\}$:

$$\frac{\partial^{|I|}}{\partial\tau^I}\xi_j(\tau)=\sum_{f^{k+1-|I|}\le i\le f^{k-|I|}-1}T'_{ij}(\tau)\xi_i(\tau), \text{ for each } |I|\ge 0 \tag{14}$$

where $T'_{ij}(\tau)$'s are holomorphic functions in τ.

Now let us look at the Taylor expansion of $\xi_j(\tau)$ at p for all $f^{k+1}\le j\le f^k-1$. We get, for $|\tau|<\epsilon$ small,

$$\xi_j(\tau)=\sum_{|I|\ge 0}\frac{\partial^{|I|}}{\partial\tau^I}\xi_j(p)\tau^I=\sum_{|I|\ge 0}\left(\sum_{f^{k+1-|I|}\le i\le f^{k-|I|}-1}T'_{ij}(p)\xi_i(p)\right)\tau^I \tag{15}$$

$$=\sum_{i\ge f^{k+1}}T_{ij}(\tau)\xi_i(p), \tag{16}$$

where if $|I|=d$ for some $d\ge 0$, then for any $f^{k+1-d}\le i\le f^{k-d}-1$, we write

$$T_{ij}(\tau)=\sum_{|I|=d}T'_{ij}(p)\tau^I.$$

Thus by taking all $n-s\le k\le s$, (16) gives us a $m\times m$ matrix $T(\tau)$, such that

$$(\xi_0(\tau),\cdots,\xi_{m-1}(\tau))=(\xi_0(p),\cdots,\xi_{m-1}(p))T(\tau) \tag{17}$$

If we set the blocks of $T(\tau)$ in the following: for each $0\le\alpha,\beta\le 2s-n$, the (α,β)-th block $T^{\alpha,\beta}$ of $T(\tau)$, is

$$T^{\alpha,\beta}=[T_{ij}(\tau)]_{f^{-\alpha+s+1}\le i\le f^{-\alpha+s}-1,\ f^{-\beta+s+1}\le j\le f^{-\beta+s}-1}, \tag{18}$$

then from (16), we see that for any $0\le\beta\le 2s-n$, and $f^{-\beta+1+s}\le j\le f^{-\beta+s}-1$, $T_{ij}(\tau)=0$ if $i\le f^{-\beta+1+s}-1$, namely if $f^{-\alpha+s+1}<f^{-\beta+s+1}$, then $T^{\alpha,\beta}=0$. As $f^k=\sum_{l=k}h^{l,n-l}$

is decreasing with respect to k, we have when $\beta > \alpha$, $T^{\alpha,\beta} = 0$. Thus we concludes that $T(\tau)$ has the following form,

$$\begin{bmatrix} T^{0,0} & 0 & 0 & \cdots & 0 & 0 & 0 \\ * & T^{1,1} & 0 & \cdots & 0 & 0 & 0 \\ * & * & T^{2,2} & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots & \vdots & \vdots \\ * & * & * & \cdots & T^{2s-n-2,2s-n-2} & 0 & 0 \\ * & * & * & \cdots & * & T^{2s-n-1,2s-n-1} & 0 \\ * & * & * & \cdots & * & * & T^{2s-n,2s-n} \end{bmatrix},$$

which means that $T(\tau)$ is a block lower triangular matrix. Moreover, as vector spaces, F_p^0 and F_q^0 are the same, thus as the transition matrix between two bases of the same vector space, $T(\tau)$ is obviously non-singular. This completes the proof of Lemma 3.6. □

Here we remark that in [8], we have used column vector for the adapted basis for the Hodge decomposition of $\Phi(p)$, while in this paper we use row vector for the adapted basis of $\Phi(p)$. Due to such different conventions, the form of matrix $T(\tau)$ in this lemma and $c(q)$ in Lemma 4.5 of [8] should be transpose to each other.

Let $\Phi : \mathcal{T} \to D$ be the period map from the Teichmüller space of polarized and marked Calabi-Yau type manifolds to the period domain. For a fixed point $p \in \mathcal{T}$, we construct the local holomorphic coordinate system $(U_p, \{\tau_1, \cdots, \tau_N\})$ around p associated to a given basis $\{\eta_0, \eta_1, \cdots, \eta_N\}$ of $H_p^{s,n-s} \oplus H_p^{s-1,n-s+1}$ as in Section 3.2. We complete the basis $\{\eta_0, \eta_1, \cdots, \eta_N\}$ to an adapted basis $\{\eta_0, \cdots, \eta_{m-1}\}$ for the Hodge decomposition of $\Phi(p)$.

In Lemma 3.6, we require $\tau \in U_p$, where U_p is a small neighbourhood of p as the Taylor expansion (16) holds for $|\tau|$ small. However, by using the connectedness of $\mathcal{T}$, we can prove the following corollary.

Corollary 3.7. *For an adapted basis $\{\zeta_0, \cdots, \zeta_{m-1}\}$ for the Hodge decomposition of $\Phi(q)$ for any $q \in \mathcal{T}$, there exists a non-singular block lower trangular matrix T such that*

$$(\zeta_0 \cdots, \zeta_{m-1}) = (\eta_0, \cdots, \eta_{m-1})T.$$

Proof. Since $\mathcal{T}$ is connected, there exists a smooth curve $\gamma(s)$ connecting $p = \gamma(0)$ and $q = \gamma(1)$ and moreover we can choose

$$0 = s_0 < s_1 < \cdots < s_{d-1} < s_d = 1,$$

such that for all $0 \leq l \leq d$, $\gamma(s_{l+1}) \in U_{\gamma(s_l)}$, with $U_{\gamma(s_l)}$ small enough.

Let $(U_{\gamma(s_l)}, \{\tau_1^{(l)}, \cdots, \tau_N^{(l)}\})$ is the local holomorphic coordinate system associated to a basis

$$\{\xi_0(\gamma(s_l)), \xi_1(\gamma(s_l)), \cdots, \xi_N(\gamma(s_l))\}$$

of $H_{\gamma(s_l)}^{s,n-s} \oplus H_{\gamma(s_l)}^{s-1,n-s+1}$ as defined in Section 3.2, such that

$$(\tau_1^{(0)}, \cdots, \tau_N^{(0)}) = (\tau_1, \cdots, \tau_N),$$

$$\{\xi_0(\gamma(s_0)), \xi_1(\gamma(s_0)), \cdots, \xi_N(\gamma(s_0))\} = \{\eta_0, \eta_1, \cdots, \eta_N\},$$
$$\{\xi_0(\gamma(s_d)), \xi_1(\gamma(s_d)), \cdots, \xi_N(\gamma(s_d))\} = \{\zeta_0, \zeta_1, \cdots, \zeta_N\}.$$

For simplicity, we will denote $\gamma(s_l) = b_l$.

For each $0 \le l \le d$, we complete the basis $\{\xi_0(b_l), \cdots, \xi_N(b_l)\}$ to an adapted basis $\{\xi_0(b_l), \cdots, \xi_{m-1}(b_l)\}$ for the Hodge decomposition of $\Phi(b_l)$, such that

$$(\xi_0(b_0), \cdots, \xi_{m-1}(b_0)) = (\eta_0, \cdots, \eta_{m-1}), \quad (\xi_0(b_d), \cdots, \xi_{m-1}(b_d)) = (\zeta_0, \cdots, \zeta_{m-1}). \tag{19}$$

Since $b_l \in U_{b_{l-1}}$, we can apply Lemma 3.6 for any $1 \le l \le d$ to get that there exists a non-singular block lower triangular matrix $T(b_l)$ such that:

$$(\xi_0(b_l), \cdots, \xi_{m-1}(b_l)) = (\xi_0(b_{l-1}), \cdots, \xi_{m-1}(b_{l-1}))T(b_l).$$

Then using this formula repeatedly, together with (19), we get that the transition matrix between the two adapted bases $\{\zeta_0, \cdots, \zeta_{m-1}\}$ and $\{\eta_0, \cdots, \eta_{m-1}\}$ for the Hodge decompositions of $\Phi(p)$ and $\Phi(q)$:

$$(\zeta_0, \cdots, \zeta_{m-1}) = (\eta_0, \cdots, \eta_{m-1})T(b_1)T(b_2) \cdots T(b_{d-1}) \cdot T(b_d).$$

Denoting $T = T(b_1)T(b_2) \cdots T(b_{d-1}) \cdot T(b_d)$, we get,

$$(\zeta_0, \cdots, \zeta_{m-1}) = (\eta_0, \cdots, \eta_{m-1})T.$$

For each $0 \le l \le d$, $T(b_l)$ is a non-singular block lower triangular $m \times m$ matrix, thus so is T. This completes the proof of the corollary. $\square$

From the proof of this corollary, one gets similar formulas as in (16) of the transformation between the adapted bases $\{\eta_0, \cdots, \eta_{m-1}\}$ and $\{\zeta_0, \cdots, \zeta_{m-1}\}$ in terms of the entries of the matrix T, i.e. for each $n - s \le k \le s$:

$$\zeta_j = \sum_{i \ge f^{k+1}} T_{ij}\eta_i, \quad \text{for } f^{k+1} \le j \le f^k - 1. \tag{20}$$

Recall that we have defined a local canonical holomorphic section $[\Omega^c_{p,\tau}]$ on U_p, and we expressed the local canonical class $[\Omega^c_{p,\tau}](q) \in H^{s,n-s}_q$ in the Taylor expansion form (12) by using the local holomorphic coordinates $(U_p, \{\tau_1, \cdots, \tau_N\})$. Using the condition that $h^{s,n-s}_p = 1$, we can show the following proposition:

Proposition 3.8. *Fix $p \in \mathcal{T}$ and $(U_p, \{\tau_1, \cdots, \tau_N\})$ the Kuranishi coordinate chart associated to a fixed basis $\{\eta_0, \eta_1, \cdots, \eta_N\}$ of $H^{s,n-s}_p \oplus H^{s-1,n-s+1}_p$, then for any $q \in U_p$, and any $[\Theta_q] \in H^{s,n-s}_q$, there exists a constant $\lambda \in \mathbb{C}$ such that*

$$[\Theta_q] = \lambda\left([\Omega^c_{p,\tau}](q)\right) = \lambda\left(\eta_0 + \sum_{i=1}^N \tau_i(q)\eta_i + g(q)\right)$$

with $g(q) \in \bigoplus_{k \ge 2} H^{s-k,n-s+k}_p$.

Proof. Since $\dim_{\mathbb{C}} H^{s,n-s}_q = 1$, and both $[\Omega^c_{p,\tau}](q)$ and $[\Theta_q]$ belong to $H^{s,n-s}_q$, there exists a complex number λ such that,

$$[\Theta_q] = \lambda([\Omega^c_{p,\tau}](q)).$$

Using (12), we get the conclusion. $\square$

Note that in the above proposition, if we let $[\Theta_q]$ vary holomorphically in q, then λ also depends on q holomorphically. But we will not need this property in this paper.

Next we will use Lemma 3.5, Corollary 3.7, and Proposition 3.8 to prove that the transition maps in the Kuranishi coordinate cover as constructed in Section 3.2 on the Teichmüller space of polarized and marked Calabi-Yau type manifolds are actually affine maps. This is given in the following theorem.

Theorem 3.9. *The Kuranishi coordinate cover on $\mathcal{T}$ is a holomorphic affine coordinate cover, thus it defines a global holomorphic affine structure on $\mathcal{T}$.*

Proof. We use the same notations as in Corollar 3.7. Let $p, q \in \mathcal{T}$, and U_p, U_q the corresponding small neighbourhoods of p and q, with $U_p \cap U_q \neq \emptyset$. Suppose that $(U_p, \{\tau_1, \cdots, \tau_N\})$ is the local holomorphic coordinate system associated to the basis $\{\eta_0, \eta_1, \cdots, \eta_N\}$ of $H_p^{s,n-s} \oplus H_p^{s-1,n-s+1}$, and $(U_q, \{\sigma_1, \cdots, \sigma_N\})$ is the local holomorphic coordinate system associated to the basis $\{\zeta_0, \zeta_1, \cdots, \zeta_N\}$ of $H_q^{s,n-s} \oplus H_q^{s-1,n-s+1}$ as defined in Section 3.2. We complete these two bases to adapted bases $\{\eta_0, \eta_1, \cdots, \eta_N, \cdots, \eta_{m-1}\}$ and $\{\zeta_0, \zeta_1, \cdots, \zeta_N, \cdots, \zeta_{m-1}\}$ for the Hodge decompositions of $\Phi(p)$ and $\Phi(q)$ respectively.

We shall prove the theorem by dividing it into the following two cases.

Case 1: When $p = q$, if $\{\eta_0, \cdots, \eta_N\}$ and $\{\zeta_0, \cdots, \zeta_N\}$ are the same, then the transition map between the coordinates is just the identity map, thus a holomorphic affine map. Otherwise, according to Lemma 3.5, the transistion map is a holomorphic affine map.

In particular, this also tells us that we can use any choice of the orthonormal basis of $H_p^{s,n-s} \oplus H_p^{s-1,n-s+1}$ around a fixed point $p \in \mathcal{T}$ for our discussion about the global holomorphic affine structure on $\mathcal{T}$.

Case 2: When $p \neq q$, for any point $r \in U_p \cap U_q$, let us compute the transition map between its coordinates in U_p and U_q which we denote by $\{\tau_1(r), \cdots, \tau_N(r)\}$ and $\{\sigma_1(r), \cdots, \sigma_N(r)\}$ respectively.

Let $[\Omega^c_{p,\tau}](r) \in H_r^{s,n-s}$ be the canonical $(s, n-s)$ class defined by (12) in the local coordinates $(U_p, \{\tau_1, \cdots, \tau_N\})$ around p,

$$[\Omega^c_{p,\tau}](r) = \eta_0 + \sum_{j=1}^{N} \tau_j(r)\eta_j + \sum_{l \geq N+1} g_l(\tau(r))\eta_l, \tag{21}$$

where $\tau(r)$ is the coordinate of r in $\{U_p, (\tau_1, \cdots, \tau_N)\}$.

Similarly, let $[\Omega^c_{q,\sigma}](r) \in H_r^{s,n-s}$ be the canonical $(s, n-s)$ class defined by (12) in the local coordinates $(U_q, \{\sigma_1, \cdots, \sigma_N\})$ around q,

$$[\Omega^c_{q,\sigma}](r) = \zeta_0 + \sum_{j=1}^{N} \sigma_j(r)\zeta_j + \sum_{l \geq N+1} f_l(\sigma(r))\zeta_l, \tag{22}$$

where $\sigma(r)$ is the coordinate of r in $\{U_p, (\sigma_1, \cdots, \sigma_N)\}$.

Since $[\Omega^c_{p,\tau}](r)$ and $[\Omega^c_{q,\sigma}](r)$ are both in $H_r^{s,n-s}$, applying Proposition 3.8, we get that there exists $\lambda \in \mathbb{C}$, such that:

$$\eta_0 + \sum_{j=1}^{N} \tau_j(r)\eta_j + \sum_{l \geq N+1} g_l(\tau(r))\eta_l = \lambda \left(\zeta_0 + \sum_{j=1}^{N} \sigma_j(r)\zeta_j + \sum_{l \geq N+1} f_l(\sigma(r))\zeta_l \right). \tag{23}$$

On the other hand, by Corollar 3.7, there exists a non-singular block lower triangular $m \times m$ matrix T, such that:

$$(\zeta_0, \cdots, \zeta_{m-1}) = (\eta_0, \cdots, \eta_{m-1})T. \tag{24}$$

According to (20), we can write (24) explicitly as follows,

$$\zeta_j = \sum_{i \geq f^{k+1}} T_{ij}\eta_i, \quad \text{for each } f^{k+1} \leq j \leq f^k - 1$$

and for each $n - s \leq k \leq s$. Thus (23) can be written as

$$\begin{aligned} &\eta_0 + \sum_{j=1}^{N} \tau_j(r)\eta_j + \sum_{l \geq N+1} g_l(\tau(r))\eta_l \\ =&\lambda \left(\sum_{i \geq 0} T_{i0}\eta_i + \sum_{j=1}^{N} \left(\sigma_j(r) \sum_{i \geq 1} T_{ij}\eta_i \right) + \sum_{l \geq N+1} \left(f_l(\sigma(r)) \sum_{j \geq N+1} T_{ij}\eta_j \right) \right). \end{aligned}$$

By comparing the coefficients of η_0, we get $\lambda = T_{00}^{-1}$. Projecting both sides of the above equation to $H_p^{s-1,n-s+1}$, we get:

$$\begin{aligned} \sum_{j=1}^{N} \tau_j(r)\eta_j &= \lambda \left(\sum_{i \geq 1} T_{i0}\eta_i + \sum_{j=1}^{N} \left(\sigma_j(r) \sum_{i \geq 1} T_{ij}\eta_i \right) \right) \\ &= \lambda \left(\sum_{j \geq 1}^{N} \left(T_{j0} + \sum_{i=1}^{N} \sigma_i(r) T_{ji} \right) \eta_j \right). \end{aligned}$$

By comparing the coefficients for each η_j, $1 \leq j \leq N$, and substituting $\lambda = T_{00}^{-1}$, we get

$$T_{00}\tau_j(r) = T_{j0} + \sum_{i=1}^{N} \sigma_i(r) T_{ji}.$$

That is,

$$T_{00} \begin{bmatrix} \tau_1(r) \\ \vdots \\ \tau_N(r) \end{bmatrix} = \begin{bmatrix} T_{10} \\ \vdots \\ T_{N0} \end{bmatrix} + \begin{bmatrix} T_{11} & \cdots & T_{1N} \\ \cdots & \cdots & \cdots \\ T_{N1} & \cdots & T_{NN} \end{bmatrix} \begin{bmatrix} \sigma_1(r) \\ \vdots \\ \sigma_N(r) \end{bmatrix}. \tag{25}$$

Since for all $0 \leq i, j \leq N$, T_{ij} are constants depending only on p and q, thus (25) shows that the transition map between $(\tau_1(r), \cdots, \tau_N(r))^T$ and $(\sigma_1(r), \cdots, \sigma_N(r))^T$ for any $r \in U_p \cap U_q$ is an affine map. This completes the proof of this theorem. $\square$

In the next section, we shall use this global affine structure on $\mathcal{T}$ to show that there exists a global affine coordinate system on the Teichmüller space of polarized and marked Calabi-Yau type manfolds, and then prove the global Torelli theorem on the Teichmüller space.

4. Global Torelli theorem on the Teichmüller space

In this section, we prove the global Torelli theorem on the Teichmüller space $\mathcal{T}$ of polarized and marked Calabi-Yau type manifolds. The proof is based on the construction of the global holomorphic flat coordinates on $\mathcal{T}$. In Section 4.1, we use the holomorphic affine structure constructed in Section 3 to prove the existence of global holomorphic flat coordinates on the Teichmüller space $\mathcal{T}$ of polarized and marked Calabi-Yau type manfolds, which induces a natural holomorphic affine embedding from the Teichmüller space into $\mathbb{C}^N \cong H_p^{s-1,n-s+1}$. In Section 4.2, we use the global holomorphic flat coordinates on $\mathcal{T}$ to prove Theorem 4.4, which is the global Torelli theorem on the Teichmüller space of polarized and marked Calabi-Yau type manifolds.

4.1. **The holomorphic affine embedding from $\mathcal{T}$ into $\mathbb{C}^N \cong H_p^{s-1,n-s+1}$.** First recall that in Section 3.2, we constructed the Kuranishi coordinate cover on the Teichmüller space $\mathcal{T}$, that is, for each $p \in \mathcal{T}$, we defined a local holomorphic coordinate chart $(U_p, \{\tau_1, \cdots, \tau_N\})$ around p which is associated to a chosen bases $\{\eta_0\}$ and $\{\eta_1, \cdots, \eta_N\}$ of $H_p^{s,n-s}$ and $H_p^{s-1,n-s+1}$ respectively. By setting $\rho_{U_p}(q) = (\tau_1(q), \cdots, \tau_N(q))^T$, for any $q \in U_p$, we then get a local holomorphic embedding around p,

$$\rho_{U_p} : U_p \to \mathbb{C}^N \cong H_p^{s-1,n-s+1} \cong T_p^{1,0}\mathcal{T}.$$

Here recall that we make the identification $H_p^{s-1,n-s+1} \cong \mathbb{C}^N$ by fixing the orthonormal basis $\{\eta_1, \cdots, \eta_N\}$ of $H_p^{s-1,n-s+1}$. Because the holomorphic affine structure on $\mathcal{T}$ is defined by the Kuranishi coordinate cover, we have that the local holomorphic coordinate map $\rho_{U_p} : U_p \to \mathbb{C}^N \cong H_p^{s-1,n-s+1}$ is an affine map on U_p. Then by applying Theorem 3.3 we get the following lemma:

Lemma 4.1. *There exists a unique global holomorphic affine extension*

$$\rho_p : \mathcal{T} \to \mathbb{C}^N \cong H_p^{s-1,n-s+1}$$

of ρ_{U_p}, such that when restricted on U_p, $\rho_p|_{U_p} = \rho_{U_p}$.

The theorem below will give us the global holomorphic affine coordinates on $\mathcal{T}$ and consequently the global holomorphic embedding of $\mathcal{T}$ into $\mathbb{C}^N \cong H_p^{s-1,n-s+1}$. Recall that in Section 3.3 we showed that the Kuranishi coordinate cover gives us a holomorphic affine structure on the Teichmüller space $\mathcal{T}$ of polarized and marked Calabi-Yau type manifolds. By the discussion in Section 3.1, we have a corresponding global holomorphic affine connection ∇ on $\mathcal{T}$. We shall use this global holomorphic affine connection ∇ and the assumption that $\mathcal{T}$ is simply connected to construct the global holomorphic coodinate functions. We then prove that it agrees with ρ_p.

Theorem 4.2. *The holomorphic affine map $\rho_p : \mathcal{T} \to \mathbb{C}^N \cong H_p^{s-1,n-s+1}$ is a holomorphic affine embedding.*

Proof. The proof is mainly to show ρ_p is given explicitly by a global holomorphic flat coordinates.

To construct the global holomorphic coordinate system $t = (t_1, \cdots, t_N)$ on $\mathcal{T}$, we start from a point $p \in \mathcal{T}$, and the local holomorphic coordinate system $(U_p, \{\tau_1, \cdots, \tau_N\})$ around p as defined in Section 3.2. Then we will follow the discussions as in page 217 of [13]. Because $(U_p, \{\tau_1, \cdots, \tau_N\})$ is a local holomorphic coordinate chart, the holomorphic

vector fields $\{\partial/\partial\tau_1, \cdots, \partial/\partial\tau_N\}$ form a local parallel frame of $T^{1,0}U_p$ with respect to the holomorphic flat connection ∇ on $\mathcal{T}$. Because $\mathcal{T}$ is simply connected and that the holomorphic connection is torsion-free and flat, we can extend this local parallel frame, by parallel transports, to a global holomorphic parallel frame $\{V_1, \cdots, V_N\}$ of $T^{1,0}\mathcal{T}$. We denote by $\{\theta_1, \cdots, \theta_N\}$ the dual frame of $\{V_1, \cdots, V_N\}$ defined by the condition

$$\theta_i(V_j) = \delta_{ij}, \quad \text{for } \ 1 \leq i, j \leq N.$$

We shall show the following two properties:

(i) $\{\theta_1, \cdots, \theta_N\}$ is a global holomorphic parallel frame of the holomorphic cotangent bundle $T^{*1,0}\mathcal{T}$;

(ii) The differential forms $\theta_1, \cdots, \theta_N$ are closed holomorphic $(1,0)$-forms on $\mathcal{T}$.

In fact, to show (i), we use the fact that the affine connection ∇ is holomorphic. Because ∇ is holomorphic, the parallel frame $\{V_1, \cdots, V_N\}$ is a global holomorphic frame of $T^{1,0}\mathcal{T}$. This implies that the dual frame $\{\theta_1, \cdots, \theta_N\}$ is a global parallel holomorphic frame of $T^{*1,0}\mathcal{T}$, which also gives that $d\theta_i$ is a $(2,0)$-form on $\mathcal{T}$, for each $1 \leq i \leq N$.

For (ii), notice the fact that ∇ is torsion-free implies that the parallel holomorphic vector fields are commutative, i.e. $[V_k, V_l] = \nabla_{V_k}V_l - \nabla_{V_l}V_k = 0$ for $1 \leq k, l \leq N$. Then we get

$$\begin{aligned} d\theta_i(V_k, V_l) =& V_k(\theta_i(V_l)) - V_l(\theta_i(V_k)) - \theta_i([V_k, V_l]) \\ =& V_k(\delta_{il}) - V_l(\delta_{ik}) - \theta_i(0) \\ =& 0, \qquad \text{for } 1 \leq i, k, l \leq N. \end{aligned}$$

This shows that the $(2,0)$ part of each $d\theta_i$ is zero. Together with the above discussion that $d\theta_i$ is a (2, 0)-form, we get that $d\theta_i = 0$ for each $1 \leq i \leq N$.

Because $\mathcal{T}$ is simply connected, these closed forms $\{\theta_1, \cdots, \theta_N\}$ are exact. This implies that there are holomorphic functions $\{t_1, \cdots, t_N\}$ uniquely defined on $\mathcal{T}$ such that

$$dt_i = \theta_i \quad \text{and} \quad t_i(p) = 0,$$

for each $1 \leq i \leq N$.

By (i), these globally defined holomorphic functions $\{t_1, \cdots, t_N\}$ satisfy that their differentials $\{dt_1, \cdots, dt_N\}$ give a global parallel frame of the holomorphic cotangent bundle of $\mathcal{T}$, hence $dt_1 \wedge \cdots \wedge dt_N$ is never zero. Therefore given a local holomorphic coordinate system around any point $q \in \mathcal{T}$, the Jacobian of the functions $\{t_1, \cdots, t_N\}$ in this local holomorphic coordinate system is nondegenerate. With this discussion, one can conclude that the functions $\{t_1, \cdots, t_N\}$ form a local holomorphic coordinate system when restricted to a holomorphic coordinate chart around q, for each $q \in \mathcal{T}$. Therefore we can conclude that $t = (t_1, \cdots, t_N)^T$ defined in this way is a holomorphic embedding from $\mathcal{T}$ to $\mathbb{C}^N$ as it forms a global holomorphic coordinate system for $\mathcal{T}$.

Next we prove that $(t_1, \cdots, t_N)^T = \rho_p$, and conclude $t = (t_1, \cdots, t_N)^T$ is an affine map since ρ_p is affine.

Notice that $dt_i = \theta_i = d\tau_i$ on U_p, then by the uniqueness of $\{t_1, \cdots, t_N\}$, we have

$$t_i(q) = \tau_i(q) \quad \text{for } \ q \in U_p, \ \ 1 \leq i \leq N.$$

By Lemma 4.1, there exists a unique global holomorphic affine map $\rho_p : \mathcal{T} \to \mathbb{C}^N \cong H_p^{s-1,n-s+1}$ that extends ρ_{U_p}, and when restricted to U_p we have

$$\rho_i(q) = \tau_i(q) \quad \text{for} \;\; q \in U_p, \;\; 1 \leq i \leq N,$$

where we write $\rho_p(q) = (\rho_1(q), \cdots, \rho_N(q))^T \in \mathbb{C}^N$. Note that for each $1 \leq i \leq N$, the coordinate function t_i and ρ_i are both globally defined holomorphic functions on $\mathcal{T}$, and they agree on an open subset $U_p \subset \mathcal{T}$, thus they must agree on the whole $\mathcal{T}$, i.e.

$$\rho_i(q) = t_i(q) \quad \text{for} \;\; q \in \mathcal{T}, \;\; 1 \leq i \leq N.$$

By combining the above discussions we conclude that the holomorphic affine map

$$\rho_p : \mathcal{T} \to \mathbb{C}^N \cong H_p^{s-1,n-s+1},$$

which can also be given by the global holomorphic flat coordinates $\rho_p = t = (t_1, \cdots, t_N)^T$, is a holomorphic affine embedding. This completes the proof of this theorem . □

Since we have obtained ρ_p by extending the local holomorphic map $\rho_{U_p} : U_p \to \mathbb{C}^N$ around p with $\rho_p(p) = 0$, we will also call these global coordinates $t = \rho_p = \{t_1, \cdots, t_N\}$ on $\mathcal{T}$ a global holomorphic flat coordinate system centered at p.

As an alternative description of the construction of this global holomorphic flat coordinate $(t_1, \cdots, t_N)$, which we only briefly describe, is by directly patching the local holomorphic coordinate charts through affine transformations to get the global holomorphic affine coordinates. Both approaches are essentially equivalent, they crucially depend on the fact that the Teichmüller space $\mathcal{T}$ of polarized and marked Calabi-Yau type manifolds is a simply connected holomorphic affine manifold. In the following construction we will use affine transformations and standard obstruction theory.

Let $\{(U_q, \varphi_q) : q \in \mathcal{T}\}$ be the Kuranishi coordinate cover on $\mathcal{T}$, where

$$\varphi_q := \rho_{U_q} = (\tau_1^q, \cdots, \tau_N^q)^T : U_q \to \mathbb{C}^N$$

denotes the local holomorphic coordinate map constructed in Section 3.2. Notice we slightly changed the notations of the local holomorphic coordinates just for making the argument here cleaner. We denote by $\mathrm{Aff}(\mathbb{C}^N)$ the group of holomorphic affine transformations from $\mathbb{C}^N$ to $\mathbb{C}^N$.

Because the Kuranishi coordinate cover on $\mathcal{T}$ is a holomorphic affine coordinate cover, we have that the transition map $\varphi_q \circ \varphi_{q'}^{-1} \in \mathrm{Aff}(\mathbb{C}^N)$ for each q and q' in $\mathcal{T}$ whenever $U_q \cap U_{q'} \neq \emptyset$. Let us fix a point $p \in \mathcal{T}$ and the holomorphic coordinate chart $\{\tau_1^p, \cdots, \tau_N^p\}$ around p. Then we can choose an $A_q \in \mathrm{Aff}(\mathbb{C}^N)$ for each coordinate chart (U_q, φ_q), such that

$$\begin{aligned} A_q \circ \varphi_q = A_{q'} \circ \varphi_{q'} \quad & \text{whenever} \;\; U_q \cap U_{q'} \neq \emptyset; \\ A_p = Id \qquad\quad & \text{on} \;\; U_p. \end{aligned}$$

By using uniqueness of the analytic continuation of holomorphic functions, or repeating the standard argument that a flat vector bundle on a simply connected space is a trivial bundle, see for examples the discussions in [11], Corollary 9.2 in Section 9 of Chapter II, and Theorem 4.1 in Section 4 of Chapter V, we know that the obstruction to make the above choices is in $\mathrm{Hom}\,(\pi_1(\mathcal{T}), \mathrm{Aff}(\mathbb{C}^N))$, where $\pi_1(\mathcal{T})$ denotes the fundamental group of

the Teichmüller space $\mathcal{T}$. Because $\mathcal{T}$ is simply connected, the obstruction vanishes. This implies that we may choose a new coordinate cover,

$$\{(U_q, \psi_q = A_q \circ \varphi_q) : \, q \in \mathcal{T}\},$$

such that

$$\psi_q \circ \psi_{q'}^{-1} = Id \quad \text{ for } \; U_q \cap U_{q'} \neq \emptyset.$$

This gives us the global holomorphic flat coordinates which agree with $\{\tau_1, \cdots, \tau_N\}$ on U_p.

To close this subsection we describe the local exponential map from $\mathbb{C}^N$ to $\mathcal{T}$, where we identify $\mathbb{C}^N$ with $T_p^{1,0}\mathcal{T}$ and $H_p^{s-1,n-s+1}$, the identification is described in Section 2.2. We shall also show that it is an inverse map of the affine embedding $\rho_{U_p} : U_p \to \mathbb{C}^N \cong H_p^{s-1,n-s+1}$.

One can define the exponential map on the holomorphic tangent space of a holomorphic affine manifold as follows. Let M be an N dimensional complex manifold with a holomorphic flat connection ∇, which is defined in Section 3.1. Then for each point $p \in M$ and a basis $\{X_1, \cdots, X_N\}$ of the holomorphic tangent space $T_p^{1,0}M$, we have the unique local holomorphic coordinate functions $\{t_1, \cdots, t_N\}$ around p, such that

$$\frac{\partial}{\partial t_i}|_p = X_i \;\text{ for each } i.$$

Definition 4.3. *Let M be a holomorphic affine manifold. For each pair $(p, \{X_1, \cdots, X_N\})$ as described above, the holomorphic map*

$$\begin{aligned} \exp : \, & T_p^{1,0}M \to M, \\ & \sum_i t_i X_i \mapsto (t_1, \cdots, t_N)^T. \end{aligned}$$

on a neighbourhood of $0 \in T_p^{1,0}M$ is called the exponential map according to $(p, \{X_1, \cdots, X_N\})$.

In particular, the Teichmuller space $\mathcal{T}$ of polarized and marked Calabi-Yau type manifold is a holomorphic affine manifold. Recall that for a given point $p \in \mathcal{T}$ and a generator $[\Omega_p]$ of $H_p^{s,n-s}$, the holomorphic tangent space $T_p^{1,0}\mathcal{T} \cong H^{0,1}(M_p, T^{1,0}M_p)$ is identified with $H_p^{s-1,n-s+1}$ by the isomorphism

$$\begin{aligned} \lrcorner : \; H^{0,1}(M_p, T^{1,0}M_p) & \to H_p^{s-1,n-s+1} \\ [\varphi] & \mapsto [\varphi \lrcorner \Omega_p]. \end{aligned}$$

If we fix bases $\{\eta_0\}$ and $\{\eta_1, \cdots, \eta_N\}$ of $H^{s,n-s}$ and $H_p^{s-1,n-s+1}$, respectively, we will have the local holomorphic coordinate chart given by the holomorphic functions $\{\tau_1, \cdots, \tau_N\}$ in a neighbourhood U_p of p as constructed in Section 3.2. Then according to the definition of holomorphic exponential map, we have

$$\exp : \, T_p^{1,0}\mathcal{T} \cong H_p^{s-1,n-s+1} \cong \mathbb{C}^N \to \mathcal{T}$$

with $\exp(\sum_{i=1}^N \tau_i \eta_i) = (\tau_1, \cdots, \tau_N)^T$, which is the holomorphic exponential map according to the point p and the basis $\{\eta_0\}$ and $\{\eta_1, \cdots, \eta_N\}$.

On the other hand, the local coordinate map of the local holomorphic coordinate system

$$\rho_{U_p}: U_p \to \mathbb{C}^N \cong H_p^{s-1,n-s+1},$$

with $\rho_p(\tau_1, \cdots, \tau_N) = \sum_{i=1}^{N} \tau_i \eta_i$ is clearly the inverse of the above holomorphic exponential map.

4.2. **Proof of the global Torelli theorem.**

Theorem 4.4. *The period map $\Phi : \mathcal{T} \to D$ is globally injective.*

Proof. For any $p \in \mathcal{T}$, we can construct a local holomorphic coordinate $(U_p, \{\tau_1, \cdots, \tau_N\})$ accociated to a fixed basis $\{\eta_0, \eta_1, \cdots, \eta_N\}$ of $H_p^{s,n-s} \oplus H_p^{s-1,n-s+1}$ respectively as in Section 3.2. We complete this basis $\{\eta_0, \cdots, \eta_N\}$ to an adapted basis $\{\eta_0, \eta_1, \cdots, \eta_N, \cdots, \eta_{m-1}\}$ for the Hodge decomposition of $\Phi(p)$.

Recall that in Section 4.1 we extend the local holomorphic coordinate map

$$\rho_{U_p} = (\tau_1, \cdots, \tau_N)^T : U_p \to \mathbb{C}^N \cong H_p^{s-1,n-s+1}$$

to a global homolomorphic flat coordinate system

$$\rho_p = (t_1, \cdots, t_N)^T : \mathcal{T} \to \mathbb{C}^N \cong H_p^{s-1,n-s+1},$$

with $(t_1(p), \cdots, t_N(p))^T = 0$, which is a global holomorphic affine embedding.

For any other point $q \in \mathcal{T}$, we have local holomorphic coordinates $(U_q, \{\sigma_1, \cdots, \sigma_N\})$ associated to a fixed Hodge basis $\{\zeta_0, \zeta_1, \cdots, \zeta_N\}$ of $H_q^{s,n-s} \oplus H_q^{s-1,n-s+1}$ as defined in Section 3.2. We can also complete the basis $\{\zeta_0, \zeta_1, \cdots, \zeta_N\}$ to an adapted basis $\{\zeta_0, \zeta_1, \cdots, \zeta_N, \cdots, \zeta_{m-1}\}$ for the Hodge decomposition $\Phi(q)$.

We can similarly extend the local holomorphic coordinate map

$$\rho_{U_q} = (\sigma_1, \cdots, \sigma_N)^T : U_q \to \mathbb{C}^N \cong H_q^{s-1,n-s+1}$$

to a global homolomorphic flat coordinate map

$$\rho_q = (t'_1, \cdots, t'_N)^T : \mathcal{T} \to \mathbb{C}^N \cong H_q^{s-1,n-s+1},$$

with $(t'_1(q), \cdots, t'_N(q))^T = 0$, which is a global holomorphic affine embedding.

Applying the similar arguments as in Lemma 3.6 and Corollary 3.7, we take a smooth curve $\gamma(s)$ to connect $p = \gamma(0)$ and $q = \gamma(1)$, and moreover we can choose

$$0 = s_0 < s_1 < \cdots < s_{d-1} < s_d = 1,$$

such that $\gamma(s_{l+1}) \in U_{\gamma(s_l)}$. For each $0 \le l \le d$, the local holomorphic coordinate chart $(U_{\gamma(s_l)}, \{\tau_1^{(l)}, \cdots, \tau_N^{(l)}\})$ is as defined in Section 3.2, which is associated to a fixed basis $\{\xi_0(\gamma(s_l)), \xi_1(\gamma(s_l)), \cdots, \xi_N(\gamma(s_l))\}$ of $H_{\gamma(s_l)}^{s,n-s} \oplus H_{\gamma(s_l)}^{s-1,n-s+1}$ such that $(\tau_1^{(0)}, \cdots, \tau_N^{(0)})^T = (\tau_1, \cdots, \tau_N)^T$, $(\tau_1^{(d)}, \cdots, \tau_N^{(d)})^T = (\sigma_1, \cdots, \sigma_N)^T$, and

$$\{\xi_0(\gamma(s_0)), \xi_1(\gamma(s_0)), \cdots, \xi_N(\gamma(s_0))\} = \{\eta_0, \eta_1, \cdots, \eta_N\},$$
$$\{\xi_0(\gamma(s_d)), \xi_1(\gamma(s_d)), \cdots, \xi_N(\gamma(s_d))\} = \{\zeta_0, \zeta_1, \cdots, \zeta_N\}.$$

For simplicity, we denote $\gamma(s_l) = b_l$.

For each local holomorphic coordinate chart $(U_{b_l}, \{\tau_1^{(l)}, \cdots, \tau_N^{(l)\}})$, we can extend the local holomorphic coordinate map

$$\rho_{U_{b_l}} = (\tau_1^{(l)}, \cdots, \tau_N^{(l)})^T : U_{b_l} \to \mathbb{C}^N \cong H_{b_l}^{s-1,n-s+1}$$

to a global holomorphic affine coordinate map,

$$\rho_{b_l} = (t_1^{(l)}, \cdots, t_N^{(l)})^T : \mathcal{T} \to \mathbb{C}^N \cong H_{b_l}^{s-1,n-s+1}$$

with $(t_1^{(l)}(b_l), \cdots, t_N^{(l)}(b_l))^T = 0$, which is a global holomorphic affine embedding.

For each $0 \leq l \leq d$, we complete the basis $\{\xi_0(b_l), \xi_1(b_l), \cdots, \xi_N(b_l)\}$ to an adapted basis $\{\xi_0(b_l), \cdots, \xi_{m-1}(b_l)\}$ for the Hodge decomposition of $\Phi(b_l)$, such that

$$(\xi_0(b_0), \cdots, \xi_{m-1}(b_0)) = (\eta_0, \cdots, \eta_{m-1}) \text{ and } (\xi_0(b_d), \cdots, \xi_{m-1}(b_d)) = (\zeta_0, \cdots, \zeta_{m-1}).$$

Since $b_l \in U_{b_{l-1}}$, we apply Lemma 3.6 for each $0 \leq l \leq d$ to get that there exists a non-singular block lower triangular matrix $T(b_l)$, such that

$$(\xi_0(b_l), \cdots, \xi_{m-1}(b_l)) = (\xi_0(b_{l-1}), \cdots, \xi_{m-1}(b_{l-1}))T(b_l). \tag{26}$$

We take $T = T(b_1)T(b_2)\cdots T(b_{d-1})T(b_d)$, which is a non-singular block lower triangular matrix, then

$$(\zeta_0, \cdots, \zeta_{m-1}) = (\eta_0, \cdots, \eta_{m-1})T.$$

Since $U_{b_l} \cap U_{b_{l-1}} \neq \emptyset$, we use the same argument as in the proof of Theorem 3.9 to get:

$$T_{00}(b_l)\begin{bmatrix}\tau_1^{(l-1)}(r)\\ \vdots \\ \tau_N^{(l-1)}(r)\end{bmatrix} = \begin{bmatrix}T_{10}(b_l)\\ \vdots \\ T_{N0}(b_l)\end{bmatrix} + \begin{bmatrix} T_{11}(b_l) & \cdots & T_{1N}(b_l) \\ \cdots & \cdots & \cdots \\ T_{N1}(b_l) & \cdots & T_{NN}(b_l)\end{bmatrix}\begin{bmatrix}\tau_1^{(l)}(r)\\ \vdots \\ \tau_N^{(l)}(r)\end{bmatrix}, \ r \in U_{b_l} \cap U_{b_{l-1}}. \tag{27}$$

Since $t_j^{(l)} = \tau_j^{(l)}$ on U_{b_l}, for all $0 \leq l \leq d, 1 \leq j \leq N$, we have for any $r \in U_{b_l} \cap U_{b_{l-1}}$:

$$T_{00}(b_l)\begin{bmatrix}t_1^{(l-1)}(r)\\ \vdots \\ t_N^{(l-1)}(r)\end{bmatrix} = \begin{bmatrix}T_{10}(b_l)\\ \vdots \\ T_{N0}(b_l)\end{bmatrix} + \begin{bmatrix} T_{11}(b_l) & \cdots & T_{1N}(b_l) \\ \cdots & \cdots & \cdots \\ T_{N1}(b_l) & \cdots & T_{NN}(b_l)\end{bmatrix}\begin{bmatrix}t_1^{(l)}(r)\\ \vdots \\ t_N^{(l)}(r)\end{bmatrix}. \tag{28}$$

Because $T_{00}(b_l)t_j^{(l)}$ is globally defined holomorphic functions on $\mathcal{T}$, so is $T_{j0}(b_l) + \sum_{i=1}^N T_{ji}(b_l)t_i^{(l)}$, for each $0 \leq l \leq d, 1 \leq j \leq N$. By (28), they agree on an open subset $U_{b_l} \cap U_{b_{l-1}} \subset \mathcal{T}$, therefore they must be the same on the whole $\mathcal{T}$, that is, for any $r \in \mathcal{T}$, (28) holds.

We now compute the transition map between t and t'. If we take $\overline{t}^{(l)} = (1, t_1^{(l)}, \cdots, t_N^{(l)})^T = (1, t^{(l)})^T$, and set the blocks of $T(b_l)$ and T as in the proof of Lemma 3.6, then (28) can be written,

$$T(b_l)^{0,0}\begin{bmatrix}1\\ t^{(l-1)}\end{bmatrix} = \begin{bmatrix} T(b_l)^{0,0} & 0 \\ T(b_l)^{1,0} & T(b_l)^{1,1}\end{bmatrix}\begin{bmatrix}1\\ t^{(l)}\end{bmatrix}. \tag{29}$$

If we take

$$\overline{T}(b_l) = \begin{bmatrix} T(b_l)^{0,0} & 0 \\ T(b_l)^{1,0} & T(b_l)^{1,1}\end{bmatrix},$$

then (29) can be written as

$$T_{00}(b_l)\overline{t}^{(l-1)} = \overline{T}(b_l)\overline{t}^{(l)}. \tag{30}$$

Thus we get the following formula for the transition map between $\overline{t}^{(0)}$ and $\overline{t}^{(d)}$,

$$T_{00}(b_1)\cdot \dots \cdot T_{00}(b_d)\overline{t}^{(0)} = \overline{T}(b_1)\cdot \dots \cdot \overline{T}(b_d)\overline{t}^{(d)}.$$

Note that since $T(b_l)$'s are all block lower triangular matrices, we have

$$T_{00}(b_1)\cdot \dots \cdot T_{00}(b_d) = T^{0,0}, \text{ and } \quad \overline{T}(b_1)\cdot \dots \cdot \overline{T}(b_d) = \begin{bmatrix} T^{0,0} & 0 \\ T^{1,0} & T^{1,1} \end{bmatrix}$$

By taking $t^{(0)} = t = (t_1, \cdots, t_N)$, and $t^{(d)} = t' = (t'_1, \cdots, t'_N)$, we get:

$$T^{0,0}(t_1, \cdots, t_N)^T = T^{1,0} + T^{1,1}(t'_1, \cdots, t'_N)^T. \tag{31}$$

Here $T^{0,0}$, $T^{1,0}$ and $T^{1,1}$ are respectively the $(0,0)$, $(1,0)$ and $(1,1)$ block of the non-singular block lower triangular matrix T.

Now we can prove $\Phi : \mathcal{T} \to D$ is injective by showing that $\Phi(p) = \Phi(q)$ implies $p = q$. Indeed, if $\Phi(p) = \Phi(q)$, then the non-singular block lower triangular matrix T is then a block diagonal matrix. In particular, we have $T^{1,0} = 0$. Without loss of generality, we can choose $\zeta_0 = \eta_0$ to get $T^{0,0} = 1$. Therefore in (31), we have

$$(t_1, \cdots, t_N)^T = T^{1,1}(t'_1, \cdots, t'_N)^T$$

with $T^{1,1}$ non-singular. Thus

$$(t_1(p), \cdots, t_N(p))^T = (0, \cdots, 0)^T = T^{1,1}(t'_1(p), \cdots, t'_N(p))^T.$$

and $T^{1,1}$ being non-singular implies that $(t'_1(p), \cdots, t'_N(p))^T = (0, \cdots, 0)^T$. But by the assumption $(t'_1(q), \cdots, t'_N(q))^T = (0, \cdots, 0)^T$ and Theorem 4.2 which tells us that $(t'_1, \cdots, t'_N)^T$ gives an embedding from $\mathcal{T}$ into $\mathbb{C}^N$, we get that $p = q$. This completes our proof. $\square$

5. Applications

In Section 5.1, based on the idea of a letter from Schmid to the authors [17], we describe a geometric way to relate the holomorphic affine structure on the Teichmüller space of polarized and marked Calabi-Yau type manifolds to the orbit N_+ in $\check{D}$ of the unipotent group. We define a Hodge completion $\overline{\mathcal{T}}$ of the Teichmüller space of polarized and marked Calabi-Yau type manifolds, and extend the period map to $\check{\Phi} : \overline{\mathcal{T}} \to N_+ \hookrightarrow \check{D}$. In Section 5.2, we further assume that the moduli space of polarized Calabi-Yau type manifolds is smooth and the global monodromy group acts on the period domain freely. Then we use the global Torelli theorem on Teichmüller space of polarized and marked Calabi-Yau type manifolds to show that the period map on the moduli space of polarized Calabi-Yau type manifolds is a covering map onto its image. As a consequence, we derive that the generic Torelli theorem on the moduli space of polarized Calabi-Yau type manifolds implies the global Torelli theorem on the moduli space of polarized Calabi-Yau type manifolds.

Again in this section, we will use the general conventions as defined in Section 2.1.

5.1. **Unipotent orbit and the Hodge completion of the Teichmüller space.** Let us first review some properties about the classifying space D for general polarized Hodge structure, and then restrict to our Calabi-Yau type manifold case. We refer the reader for more details to Section 2 and Section 3 in [16].

The orthogonal group of the bilinear form Q in the definition of Hodge structure is a linear algebraic group, defined over $\mathbb{Q}$. The group of the $\mathbb{C}$-rational points is,

$$G_{\mathbb{C}} = \{g \in GL(H_{\mathbb{C}})|\ Q(gu, gv) = Q(u, v) \text{ for all } u, v \in H_{\mathbb{C}}\}.$$

which acts on $\check{D}$. As one can check in [5] that $G_{\mathbb{C}}$ acts transitively on $\check{D}$.

The group of real points in $G_{\mathbb{C}}$ is

$$G_{\mathbb{R}} = \{g \in GL(H_{\mathbb{R}})|\ Q(gu, gv) = Q(u, v) \text{ for all } u, v \in H_{\mathbb{R}}\}.$$

which acts as a group of automorphisms on D. Again one can use simple arguments in linear algebra to conclude that $G_{\mathbb{R}}$ acts transitively on D.

Consider a variation of Hodge structure $\Phi : S \to D$ from a complex manifold S to the classifying space as defined in Section 2.1. Let us fix a point $p \in S$, with the image $o =: \Phi(p) = \{F_p^k\}_{k=0}^n \in D$. The points $p \in S$ and $o \in D \subset \check{D}$ may be referred as the base points or reference points. We also fixed an adapted basis $\{\eta_0, \cdots, \eta_{m-1}\}$ for the Hodge decomposition of $\Phi(p)$. With the fixed adapted basis $\{\eta_0, \cdots, \eta_{m-1}\}$ at the base point, one obtains matrix representations for elements in $G_{\mathbb{C}}$ and $G_{\mathbb{R}}$.

First, let us exhibit $\check{D}$ and D as quotients of $G_{\mathbb{C}}$ and $G_{\mathbb{R}}$. Notice that a linear transformation $g \in G_{\mathbb{C}}$ preserves the reference point precisely when $gF_p^k = F_p^k$. Thus this gives the identification

$$\check{D} \simeq G_{\mathbb{C}}/B, \quad \text{with} \quad B = \{g \in G_{\mathbb{C}}|\ gF_p^k = F_p^k, \text{ for any } k\}. \tag{32}$$

Notice that this isomorphism is a complex holomorphic isomorphism. In fact, with the identification (32) the identity coset and the base point o correspond to each other. As a quotient of a complex Lie group by a closed complex Lie subgroup, $G_{\mathbb{C}}/B$ has the structure of complex manifold.

Similarly, one obtains an analogous identification

$$D \simeq G_{\mathbb{R}}/V \hookrightarrow \check{D}, \text{ with } V = G_{\mathbb{R}} \cap B,$$

where the embedding corresponds to the inclusion $G_{\mathbb{R}}/V = G_{\mathbb{R}}/G_{\mathbb{R}} \cap B \subset G_{\mathbb{C}}/B$.

Next let us express the holomorphic tangent space of $\check{D}$ and D as quotients of Lie algebras.

The Lie algebra $\mathfrak{g}$ of the complex Lie group $G_{\mathbb{C}}$ can be described as

$$\begin{aligned}\mathfrak{g} &= \{X \in \mathrm{End}(H_{\mathbb{C}})|\ Q(Xu, v) + Q(u, Xv) = 0, \text{ for all } u, v \in H_{\mathbb{C}}\} \\ &= \{X \in \mathrm{End}(H_{\mathbb{C}})|\ XQ + QX^T = 0\}.\end{aligned}$$

It is a simple complex Lie algebra, which contains $\mathfrak{g}_0 = \{X \in \mathfrak{g}|\ XH_{\mathbb{R}} \subset H_{\mathbb{R}}\}$ as a real form, i.e. $\mathfrak{g} = \mathfrak{g}_0 \oplus i\mathfrak{g}_0$. With the inclusion $G_{\mathbb{R}} \subset G_{\mathbb{C}}$, $\mathfrak{g}_0$ becomes Lie algebra of $G_{\mathbb{R}}$. One observes that the reference Hodge structure $\{H_p^{k,n-k}\}_{k=0}^n$ of $H_{\mathbb{C}}$ induces a Hodge structure of weight zero on $\mathrm{End}(H_{\mathbb{C}})$. It can be checked that the subspace $\mathfrak{g} \subset \mathrm{End}(H_{\mathbb{C}})$ carries a sub-Hodge structure of weight zero, namely,

$$\mathfrak{g} = \bigoplus_{k\in\mathbb{Z}} \mathfrak{g}^{k,-k}, \text{ with } \mathfrak{g}^{k,-k} = \{X \in \mathfrak{g}|XH_p^{r,n-r} \subset H_p^{r+k,n-r-k}\}. \tag{33}$$

Since the Lie algebra of B consists of those $X \in \mathfrak{g}$ that preserves the reference Hodge filtration $\{F_p^k\}_{k=0}^n$, according to the decomposition (33), one has

$$\mathfrak{b} = \bigoplus_{k \geq 0} \mathfrak{g}^{k,-k}. \tag{34}$$

The Lie algebra $\mathfrak{v}$ of V is thus

$$\mathfrak{v} = \mathfrak{g}_0 \cap \mathfrak{b} = \mathfrak{g}_0 \cap \mathfrak{b} \cap \overline{\mathfrak{b}} = \mathfrak{g}_0 \cap \mathfrak{g}^{0,0}. \tag{35}$$

As an observation, we notice that

$$\mathrm{Ad}(g)(\mathfrak{g}^{k,-k}) \subset \bigoplus_{i \geq k} \mathfrak{g}^{i,-i}, \quad \text{for } g \in B. \tag{36}$$

In terms of matrix representation with the fixed base point and fixed adapted basis at the base point, elements in $B \subset G_{\mathbb{C}}$ are of non-singular block upper triangular, and elements in $V \subset G_{\mathbb{R}}$ are of non-singular block diagonal. From (34), one can see that any element in $\mathfrak{b} \subset \mathfrak{g}$ is a block upper triangular matrix. Similarly from (35), elements in $\mathfrak{v} \subset \mathfrak{g}_0$ are block diagonal matrices.

With the above discussion, one gets that the holomorphic tangent space of $\check{D} \cong G_{\mathbb{C}}/B$ at p is naturally isomorphic to $\mathfrak{g}/\mathfrak{b}$, i.e.

$$T_o^{1,0}\check{D} \cong \mathfrak{g}/\mathfrak{b}. \tag{37}$$

Because of (36),

$$\mathfrak{b} \oplus \mathfrak{g}^{-1,1}/\mathfrak{b} \subset \mathfrak{g}/\mathfrak{b} \tag{38}$$

defines an $\mathrm{Ad}(B)$-invariant subspace. By left translation via $G_{\mathbb{C}}$, it gives rise to a $G_{\mathbb{C}}$-invariant holomorphic subbundle of the holomorphic tangent bundle $T_o^{1,0}\check{D}$ at the base point. It will be denoted by $T_{o,h}^{1,0}\check{D}$, and will be referred to as holomorphic horizontal tangent bundle at the base point. One can check that this construction does not depend on the choice of the base point $o \in D$.

As another interpretation of this holomorphic horizontal bundle at the base point $o \in D$, one has:

$$T_{o,h}^{1,0}\check{D} \simeq T_o^{1,0}\check{D} \cap \bigoplus_{k \geq 1} \mathrm{Hom}(F_p^k/F_p^{k-1}, F_p^{k-1}/F_p^{k-2}). \tag{39}$$

The horizontal tangent subbundle at the reference point o, restricted to D, determines a subbundle $T_{o,h}^{1,0}D$ of the holomorphic tangent bundle $T_o^{1,0}D$ of D at the reference point. The $G_{\mathbb{C}}$-invariace of $T_{o,h}^{1,0}\check{D}$ implies the $G_{\mathbb{R}}$-invariance of $T_{o,h}^{1,0}D$.

Now let us take a nilpotent Lie subalgebra of $\mathfrak{g}$:

$$\mathfrak{n}_+ = \bigoplus_{k \geq 1} \mathfrak{g}^{-k,k}$$

with its corresponding unipotent Lie subgroup in $G_{\mathbb{C}}$,

$$N_+ = \exp(\mathfrak{n}_+).$$

With the above descriptions (33) and (34) of the $\mathfrak{g}$ and $\mathfrak{b}$ respectively, one gets

$$\mathfrak{g}/\mathfrak{b} \cong \mathfrak{n}_+. \tag{40}$$

In terms of matrix representation with the fixed base point and fixed adapted basis at the base point, the elements in $\mathfrak{n}_+$ are block lower triangular matrices in $\mathfrak{g}$ with the diagonal blocks being all zeros. Correspondingly, elements in N_+ are block lower triangular matrices in $G_{\mathbb{C}}$ with the diagonal blocks being all identities.

Note that N_+ is defined to be a subgroup in $G_{\mathbb{C}}$, but we can also view N_+ as a subset in $\check{D}$ as follows:

$$N_+ = N_+(\text{ base point }) \cong N_+B/B \subset \check{D},$$

that is, we identify an element $c \in N_+$ with $[c] = cB \in \check{D}$. Without causing confusion, we denote by N_+ the unipotent orbit of the base point under the action of the unipotent group N_+. In the rest of the paper, we will view N_+ as a subset of $\check{D}$ in this way.

Using the identification (37), (38) and (40), and the property that $D \subset \check{D}$ open, we get the following isomorphisms and an affine embedding:

$$T^{1,0}_{o,h}D = T^{1,0}_{o,h}\check{D} \cong \mathfrak{b} \oplus \mathfrak{g}^{-1,1}/\mathfrak{b} \hookrightarrow \mathfrak{g}/\mathfrak{b} \cong \mathfrak{n}_+.$$

In short, we get the following affine embedding,

$$T^{1,0}_{o,h}D \hookrightarrow \mathfrak{n}_+.$$

We now restrict our discussions to the case of Calabi-Yau type manifolds.

By the interpretation (39), as well as the Griffiths transversality, we know that for the period map $\Phi : \mathcal{T} \to D$ from the Teichmüller space of polarized and marked Calabi-Yau type manifolds to the period domain, we have

$$\Phi_*(T^{1,0}_p\mathcal{T}) \subset T^{1,0}_{o,h}D$$

at the base point. Moreover, together with Proposition 2.6, we have the following isomorphisms and affine embeddings,

$$H^{s-1,n-s+1}_p \cong P^s_p \circ \Phi_*(T^{1,0}_p\mathcal{T}) \hookrightarrow \Phi_*(T^{1,0}_p\mathcal{T}) \subset T^{1,0}_{o,h}D \cong \mathfrak{b} \oplus \mathfrak{g}^{-1,1}/\mathfrak{b} \hookrightarrow \mathfrak{g}/\mathfrak{b} \cong \mathfrak{n}_+. \tag{41}$$

For the base point $p \in \mathcal{T}$, we can define a local holomorphic coordinate chart $(U_p, \tau_1, \cdots, \tau_N)$ around p as in Section 3.2 associated to the fixed basis $\{\eta_0, \cdots, \eta_N\}$ of $H^{s,n-s}_p \oplus H^{s-1,n-s+1}_p$. We then get a local holomorphic affine embedding

$$\rho_{U_p} : U_p \to \mathbb{C}^N \cong H^{s-1,n-s+1}_p \cong T^{1,0}_p\mathcal{T} \tag{42}$$

by letting $\rho_{U_p}(q) = (\tau_1(q), \cdots, \tau_N(q))$ for any $q \in U_p$. Combining (41) and (42), we have the following inclusion

$$U_p \xhookrightarrow{\rho_{U_p}} \mathbb{C}^N \xhookrightarrow{\varphi} \mathfrak{n}_+ .$$

We shall define a map $\psi_{U_p} : U_p \to N_+$, and use the matrix representation to show that the following diagram is commutative:

$$\begin{tikzcd}
U_p \arrow[r, hook, "\rho_{U_p}"] \arrow[d, "\Phi"] \arrow[dr] & \mathbb{C}^N \arrow[r, hook, "\varphi"] \arrow[d, "\psi_{U_p}"] & \mathfrak{n}_+ \arrow[dl, "\exp"] \\
D \arrow[dr] & N_+ \arrow[d, "i"] & \\
& \check{D}. &
\end{tikzcd}$$

First, we can choose any adapted basis $\{\zeta_0, \cdots, \zeta_{m-1}\}$ for the Hodge decomposition of $\Phi(q)$, and by Lemma 3.6, we know that there exists a non-singular block lower triangular matrix $T(q)$, such that

$$(\zeta_0, \cdots, \zeta_{m-1}) = (\eta_0, \cdots, \eta_{m-1})T(q).$$

More explicitly, if we denote $\gamma = 2s - n$, then $T(q)$ has the following form,

$$\begin{bmatrix} T^{0,0} & 0 & 0 & \cdots & 0 & 0 & 0 \\ T^{1,0} & T^{1,1} & 0 & \cdots & 0 & 0 & 0 \\ T^{2,0} & T^{2,1} & T^{2,2} & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots & \vdots & \vdots \\ T^{\gamma-2,0} & T^{\gamma-2,1} & T^{\gamma-2,2} & \cdots & T^{\gamma-2,\gamma-2} & 0 & 0 \\ T^{\gamma-1,0} & T^{\gamma-1,1} & T^{\gamma-1,2} & \cdots & T^{\gamma-1,\gamma-2} & T^{\gamma-1,\gamma-1} & 0 \\ T^{\gamma,0} & T^{\gamma,1} & T^{\gamma,2} & \cdots & T^{\gamma,\gamma-2} & T^{\gamma,\gamma-1} & T^{\gamma,\gamma} \end{bmatrix}.$$

In terms of matrix representation, we let W denote the set of all block lower triangular matrices in $G_{\mathbb{C}}$, then one can see that $T(q) \in W$. Moreover, since $\Phi(q) \in D \cong G_{\mathbb{R}}/V$, thus $[T(q)] \in D$, i.e. there exists $X \in B$, such that $T(q)X \in G_{\mathbb{R}}$. Therefore we can firstly define

$$\widetilde{\psi}_{U_p} : U_p \to W, \quad q \mapsto T(q).$$

and then we can define $j : W \to N_+$ by sending $T(q)$ to the following element in N_+,

$$j(T(q)) = \begin{bmatrix} I & 0 & \cdots & 0 & 0 \\ (T^{0,0})^{-1}T^{1,0} & I & \cdots & 0 & 0 \\ \vdots & \vdots & \cdots & \vdots & \vdots \\ (T^{0,0})^{-1}T^{\gamma-1,0} & (T^{1,1})^{-1}T^{\gamma-1,1} & \cdots & I & 0 \\ (T^{0,0})^{-1}T^{\gamma,0} & (T^{1,1})^{-1}T^{\gamma,1} & \cdots & (T^{,\gamma-1,\gamma-1})^{-1}T^{\gamma,\gamma-1} & I \end{bmatrix} \in N_+.$$

Actually, the map j is simply the quotient map $W \to W/W \cap B$.

Now we can define ψ_{U_p} as the composition map:

$$\psi_{U_p} = j \circ \widetilde{\psi}_{U_p} : U_p \to N_+.$$

With the above description about ψ_{U_p}, one can easily see that the above local diagram is commutative. In particular,

$$\rho_{U_p} : U_p \to \mathbb{C}^N, \quad q \mapsto (T^{0,0})^{-1}T^{1,0}.$$

We make a remark about the choices of the adapted bases for $\Phi(q)$. If we choose different adapted basis $(\zeta'_0, \cdots, \zeta'_{m-1})$ for the Hodge decomposition of $\Phi(q)$, then we still get a non-singular block lower triangular matrix $T(q)'$, such that $(\zeta'_0, \cdots, \zeta'_{m-1}) = (\eta_0, \cdots, \eta_{m-1})T(q)'$. As another adapted basis of the Hodge decomposition for $\Phi(q)$, there exists $C \in V$, such that $(\zeta'_0, \cdots, \zeta'_{m-1}) = (\zeta_0, \cdots, \zeta_{m-1})C$, which gives $(\zeta_0, \cdots, \zeta_{m-1})C = (\eta_0, \cdots, \eta_{m-1})T(q)'$. On the other hand, $(\zeta_0, \cdots, \zeta_{m-1}) = (\eta_0, \cdots, \eta_{m-1})T(q)$, therefore $T(q)'C^{-1} = T(q)$. Thus we conclude that as an element in $D \cong G_{\mathbb{R}}/V$, $[T(q)'C^{-1}] = [T(q)] \in D$. Therefore since we view $N_+ \subset \check{D}$, the map $\psi_{U_p} : U_q \to N_+$ doesn't depend on the choices of the adapted bases for the Hodge decomposition of $\Phi(q)$.

Now let us extend the local commutative diagram to a global commutative diagram.

On one hand, recall that in Section 4.1, we extend ρ_{U_p} to a global holomorphic affine map,

$$\rho_p : \mathcal{T} \to \mathbb{C}^N \cong H_p^{s-1,n-s+1}. \tag{43}$$

Now combining (41), (42) and (43), we have a global affine embedding:

$$\begin{tikzcd} \mathcal{T} \arrow[r, hook, "\rho_p"] & \mathbb{C}^N \arrow[r, hook, "\varphi"] & \mathfrak{n}_+ \end{tikzcd}$$

with the affine structure on $\mathcal{T}$ defined in Section 3.

On the other hand we can also extend $\psi_{U_p} : U_p \to N_+$, in the following way. For any point $q \in \mathcal{T}$, we choose an adapted basis $\{\zeta_0, \cdots, \zeta_{m-1}\}$ for the Hodge decomposition for $\Phi(q)$, then by Corollary 3.7, we have a non-singular block lower triangular matrix $T(q)$, such that

$$(\zeta_0, \cdots, \zeta_{m-1}) = (\eta_0, \cdots, \eta_{m-1})T(q).$$

Therefore, using exactly the same definition as for $\psi_{U_p} : U_p \to N_+$, we can define a global affine map

$$\psi_p : \mathcal{T} \to N_+.$$

Moreover with the defnition of this ψ_p, we can define

$$\rho'_p : \mathcal{T} \to \mathbb{C}^N, \quad q \mapsto (T^{0,0})^{-1}T^{1,0}.$$

If we write $\rho_p = (\rho_1, \cdots, \rho_N)$ and $\rho'_p = (\rho'_1, \cdots, \rho'_N)$, obviously by their definitions, we have

$$\rho_i = \rho'_i \quad \text{on } U_p,$$

with $U_p \subset \mathcal{T}$ open. As globally defined holomorphic functions, they must agree on the whole $\mathcal{T}$, i.e.

$$\rho_i = \rho'_i \quad \text{on } \mathcal{T}.$$

Therefore we get the global commutative diagram:

$$\begin{tikzcd} \mathcal{T} \arrow[r, hook, "\rho_p"] \arrow[d, "\Phi"] \arrow[dr, "\psi_p"] & \mathbb{C}^N \arrow[r, hook, "\varphi"] \arrow[d] & \mathfrak{n}_+ \arrow[dl, "\exp"] \\ D \arrow[dr] & N_+ \arrow[d, "i"] & \\ & \check{D}. & \end{tikzcd}$$

From this diagram, we can extend the period map on the Techmülcer space of Calabi-Yau type manifolds to the following map:

$$\check{\Phi} : \mathbb{C}^N \to N_+ \hookrightarrow \check{D}.$$

Take $\overline{\mathcal{T}} = \check{\Phi}^{-1}(D)$, and we call it the *Hodge completion* of the Teichmüller space $\mathcal{T}$ of polarized and marked Calabi-Yau type manifolds. Notice that $\overline{\mathcal{T}} \supset \mathcal{T}$. Indeed, because

$\check{\Phi}|_{\mathcal{T}} = \Phi|_{\mathcal{T}}$, we get $\check{\Phi}(\mathcal{T}) = \Phi(\mathcal{T}) \subset D$. Therefore $\check{\Phi}^{-1}(D) = \overline{\mathcal{T}} \supset \mathcal{T}$. As a restriction of the affine embedding $\varphi : \mathbb{C}^N \to \mathfrak{n}_+$ on the $\overline{\mathcal{T}}$, we get the following affine embedding,

$$\overline{\varphi} = \varphi|_{\overline{\mathcal{T}}} : \overline{\mathcal{T}} \hookrightarrow \mathfrak{n}_+ \cong \mathbb{C}^d$$

with $d = \dim \mathfrak{n}_+$, as well as a map on $\overline{\mathcal{T}}$:

$$\check{\Phi} : \overline{\mathcal{T}} \to N_+ \hookrightarrow \check{D},$$

which is an extension of the period map Φ.

In a subsequent paper we will study more detailed about the geometric properties of this Hodge completion $\overline{\mathcal{T}}$ of the Teichmüller space $\mathcal{T}$ of polarized and marked Calabi-Yau type manifolds.

5.2. **The period map on the moduli space.** In this subsection, we assume that the moduli space of polarized Calabi-Yau type manifolds is smooth. Let $G_{\mathbb{Z}} = \{g \in G_{\mathbb{C}} |\ gH_{\mathbb{Z}} \subset H_{\mathbb{Z}}\}$, then for the smooth moduli space $\mathcal{M}$, we can consider the period map

$$\Phi_{\mathcal{M}} : \mathcal{M} \to D/\Gamma,$$

where $\Gamma = \mathrm{Image}(\pi_1(\mathcal{M}) \to G_{\mathbb{Z}})$ denotes the global monodromy group which acts properly and discontinuously on the period domain D.

For the two period maps Φ and $\Phi_{\mathcal{M}}$, we have the following commutative diagram,

$$\begin{tikzcd} \mathcal{T} \arrow[r, hook, "\Phi"] \arrow[d, "\pi_{\mathcal{T}}"] \arrow[dr, "\widetilde{\Phi}"] & D \arrow[d, "\pi_D"] \\ \mathcal{M} \arrow[r, "\Phi_{\mathcal{M}}"] & D/\Gamma, \end{tikzcd}$$

where $\pi_{\mathcal{T}} :\ \mathcal{T} \to \mathcal{M}$ is a covering map and $\widetilde{\Phi} = \pi_D \circ \Phi$. The image of the period map $\Phi_{\mathcal{M}}$ is an analytic subvariety of D/Γ. We refer the reader to page 156 of [6] for details of the analyticity of the image of the period mapping.

Global Torelli problem on the moduli space $\mathcal{M}$ asks when $\Phi_{\mathcal{M}}$ is injective, and generic Torelli problem asks when there exists an open dense subset $U \subset \mathcal{M}$ such that $\Phi_{\mathcal{M}}|_U$ is injective. In both cases we need to understand the global Torelli property of the period map $\Phi_{\mathcal{M}}$ on the moduli space.

Furthermore, we assume Γ acts on D freely, thus the quotient space D/Γ is a smooth analytic variety. Therefore the quotient map

$$\pi_D :\ D \to D/\Gamma$$

is a covering map. We shall show in the following theorem that $\Phi_{\mathcal{M}}$ is a covering map onto its image, and then conclude that generic Torelli theorem implies global Torelli theorem on the moduli space of polarized Calabi-Yau type manifolds.

Theorem 5.1. *Let $\mathcal{M}$ be the moduli space of polarized Calabi-Yau type manifolds. If $\mathcal{M}$ is smooth, and the global monodormy group Γ acts on D freely, then the period map on the moduli space $\Phi_{\mathcal{M}} : \mathcal{M} \to D/\Gamma$ is a covering map from $\mathcal{M}$ to its image in D/Γ. As a consequence, if the period map on the moduli space $\Phi_{\mathcal{M}}$ is generically injective, then it is globally injective.*

Proof. First, we show that $\Phi_{\mathcal{M}}$ is a covering map from $\mathcal{M}$ to its image in D/Γ. This relies on the following lemma,

Lemma 5.2. *Let $\widetilde{\Phi}: \mathcal{T} \to D \to D/\Gamma$ be the composition of Φ and the covering map $D \to D/\Gamma$. If there exist two points $\widetilde{p}_1 \neq \widetilde{p}_2 \in \mathcal{T}$ such that $\widetilde{\Phi}(\widetilde{p}_1) = \widetilde{\Phi}(\widetilde{p}_2)$, then there exist $\widetilde{V}_1$ and $\widetilde{V}_2$, which are neighbourhoods of $\widetilde{p}_1$ and $\widetilde{p}_2$ respectively, such that $\widetilde{\Phi}(\widetilde{V}_1) = \widetilde{\Phi}(\widetilde{V}_2)$, $\widetilde{V}_1 \cap \widetilde{V}_2 = \emptyset$, and the map*

$$\widetilde{\Phi}: \widetilde{V}_i \to \widetilde{\Phi}(\widetilde{V}_i)$$

for each $i = 1, 2$ is an isomorphism.

Proof of Lemma 5.2. By the argument at the begin of Section 2.1, we can identify a point $\Phi(p) = \{F_p^s \subset F_p^{s-1} \subset \cdots \subset F_p^{n-s}\} \in D$ with its the Hodge decomposition $\Phi(p) = \{H_p^{k,n-k}\}_{k=n-s}^s$.

Let $\Phi(\widetilde{p}_1) = \{H_{\widetilde{p}_1}^{k,n-k}\}_{k=n-s}^s$ and $\Phi(\widetilde{p}_2) = \{H_{\widetilde{p}_2}^{k,n-k}\}_{k=n-s}^s$ be the corresponding Hodge decompositions. Then the condition $\widetilde{\Phi}(\widetilde{p}_1) = \widetilde{\Phi}(\widetilde{p}_2)$ implies that there exists some $\gamma \in \Gamma \subset \mathrm{Aut}(H^n(M, \mathbb{Z}))$, such that $\gamma \cdot \Phi(\widetilde{p}_1) = \Phi(\widetilde{p}_2)$.

We fix an adapted basis $\{\eta_0^{(1)}, \eta_1^{(1)}, \cdots, \eta_N^{(1)}, \cdots, \eta_{m-1}^{(1)}\}$ for the Hodge decomposition of $\Phi(\widetilde{p}_1)$. Let

$$\{\eta_0^{(2)}, \eta_1^{(2)}, \cdots, \eta_N^{(2)}, \cdots, \eta_{m-1}^{(2)}\} = \{\gamma \cdot \eta_0^{(1)}, \gamma \cdot \eta_1^{(1)}, \cdots, \gamma \cdot \eta_N^{(1)}, \cdots, \gamma \cdot \eta_{m-1}^{(1)}\}.$$

Then $\{\eta_0^{(2)}, \eta_1^{(2)}, \cdots, \eta_N^{(2)}, \cdots, \eta_{m-1}^{(2)}\}$ forms an adapted basis for the Hodge decomposition of $\Phi(\widetilde{p}_2)$.

Let $(U_{\widetilde{p}_i}, \{\tau_1^{(i)}, \cdots, \tau_N^{(i)})\})$ be the local holomorphic coordinate chart associated to the basis $\{\eta_0^{(i)}, \eta_1^{(i)}, \cdots, \eta_N^{(i)}\}$ of $H_{\widetilde{p}_i}^{s,n-s} \oplus H_{\widetilde{p}_i}^{s-1,n-s+1}$, for $i = 1, 2$ respectively, which are defined in Section 3.2. In this proof, we use the same notation for a point $\tau^{(i)} \in U_{\widetilde{p}_i}$ as its corresponding coordinate in the local holomorphic coordinate $(U_{\widetilde{p}_i}, \{\tau_1^{(i)}, \cdots, \tau_N^{(i)}\})$. Then Theorem 4.2 shows that we have the following embeddings,

$$\rho_1: U_{\widetilde{p}_1} \to \mathbb{C}^N \cong H_{\widetilde{p}_1}^{s-1,n-s+1}, \quad \rho_2: U_{\widetilde{p}_2} \to \mathbb{C}^N \cong H_{\widetilde{p}_2}^{s-1,n-s+1}.$$

with $\rho_i(\tau_1^{(i)}, \cdots, \tau_N^{(i)}) = \sum_{j=1}^N \tau_j^{(i)} \eta_j^{(i)}$, for $i = 1, 2$.

Because $\dim_{\mathbb{C}} U_{\widetilde{p}_i} = \dim_{\mathbb{C}} H_{\widetilde{p}_i}^{s-1,n-s+1} = \dim_{\mathbb{C}} \mathcal{T}$ and that ρ_i is an embedding, we have that the image $\rho_i(U_{\widetilde{p}_i})$ is open in $H_{\widetilde{p}_i}^{s-1,n-s+1}$ for each $i = 1, 2$. This implies that $\gamma \cdot \rho_1(U_{\widetilde{p}_1})$ and $(\gamma \cdot \rho_1(U_{\widetilde{p}_1})) \cap \rho_2(U_{\widetilde{p}_2})$ are also open in $H_{\widetilde{p}_2}^{s-1,n-s+1}$. Together with the fact that $\gamma \cdot \rho_1(\widetilde{p}_1) = \rho_2(\widetilde{p}_2) \in (\gamma \cdot \rho_1(U_{\widetilde{p}_1})) \cap \rho_2(U_{\widetilde{p}_2}) \neq \emptyset$, we get that there exists a neighbourhood W of $\rho_2(\widetilde{p}_2)$ in $H_{\widetilde{p}_2}^{s-1,n-s+1}$, such that

$$W \subset (\gamma \cdot \rho_1(U_{\widetilde{p}_1})) \cap \rho_2(U_{\widetilde{p}_2}).$$

Let $\widetilde{V}_1 = \rho_1^{-1}(\gamma^{-1} \cdot W) \subset U_{\widetilde{p}_1}$ and $\widetilde{V}_2 = \rho_2^{-1}(W) \subset U_{\widetilde{p}_2}$, then the restriction maps

$$\rho_1|_{\widetilde{V}_1}: \widetilde{V}_1 \to W, \tag{44}$$

$$\rho_2|_{\widetilde{V}_2}: \widetilde{V}_2 \to W, \tag{45}$$

are biholomorphic maps since the ρ_1 and ρ_2 are embeddings. Then from the global commutative diagram introduced in Section 5.1, we get the following composition maps,

$$\gamma \cdot \Phi_1|_{\widetilde{V}_1}: \widetilde{V}_1 \cong W \hookrightarrow \mathbb{C}^N \cong H_{\widetilde{p}_2}^{s-1,n-s+1} \hookrightarrow \mathfrak{n}_+ \xrightarrow{\exp} N_+ \longrightarrow \check{D},$$

$$\Phi_2|_{\widetilde{V_2}} : \widetilde{V_2} \cong W \hookrightarrow \mathbb{C}^N \cong H^{s-1,n-s+1}_{\widetilde{p_2}} \hookrightarrow \mathfrak{n}_+ \xrightarrow{\exp} N_+ \longrightarrow \check{D}.$$

Notice that the composition maps from W to $\check{D}$ in the above two maps are the same. Together with the isomorphisms between $\widetilde{V_1} \cong W$ and $\widetilde{V_2} \cong W$ as defined in (44) and (45), We now can conclude that $\gamma \cdot \Phi(\widetilde{V_1}) = \Phi(\widetilde{V_2})$, which implies $\widetilde{\Phi}(\widetilde{V_1}) = \widetilde{\Phi}(\widetilde{V_2}) = \pi_D(W)$. By shrinking W properly, we can make $\widetilde{V_1}$ and $\widetilde{V_2}$ disjoint, and also we have that the map

$$\widetilde{\Phi} : \widetilde{V_i} \xrightarrow{\Phi} W \xrightarrow{\pi_D} \widetilde{\Phi}(\widetilde{V_i})$$

for each $i = 1, 2$ is an isomorphism. □

Notice that in the following commutative diagram:

$$\begin{tikzcd}
\mathcal{T} \arrow[r, hook, "\Phi"] \arrow[d, "\pi_{\mathcal{T}}"] \arrow[dr, "\widetilde{\Phi}"] & D \arrow[d, "\pi_D"] \\
\mathcal{M} \arrow[r, "\Phi_{\mathcal{M}}"] & D/\Gamma,
\end{tikzcd}$$

since $\mathcal{T}$ and $\mathcal{M}$ are both smooth, the map $\pi_{\mathcal{T}}$ is a covering map. This implies that $\pi_{\mathcal{T}}$ is a local isomorphism.

Now for any Hodge structure $\{H^{k,n-k}\}_{k=n-s}^{s} \in D/\Gamma$, if the pre-image $\Phi_{\mathcal{M}}^{-1}(\{H^{k,n-k}\}_{k=n-s}^{s}) = \{p_i | \ i \in I\}$ is not empty. Take $\{\widetilde{p_j} | j \in J\} = \pi_{\mathcal{T}}^{-1}(\{p_i \mid i \in I\})$. By Lemma 5.2, for each $j \in J$, we get $\widetilde{V_j} \subset \mathcal{T}$ which is a neighbourhood around $\widetilde{p_i}$ such that

$$\widetilde{\Phi} : \widetilde{V_j} \to \widetilde{\Phi}(\widetilde{V_j})$$

is an isomorphism, $\cup_j \widetilde{V_j}$ is a disjoint union, and all the $\widetilde{\Phi}(\widetilde{V_j})$ are the same. Now take $\{V_k | \ k \in K\} = \{\pi_{\mathcal{T}}(\widetilde{V_j}) | \ j \in J\}$. By shrinking the set W as in Lemma 5.2 properly, we can still make $\cup_k V_k$ a disjoint union, and the images $\Phi_{\mathcal{M}}(V_k) = \pi_D(W)$, for any k, are still all the same. Moreover, since $\pi_{\mathcal{T}}$ is covering map, the map

$$\Phi_{\mathcal{M}} : V_k \to \Phi_{\mathcal{M}}(V_k)$$

is still an isomorphism for each $k \in K$. Therefore the holomorphic map

$$\Phi_{\mathcal{M}} : \ \mathcal{M} \to \Phi_{\mathcal{M}}(\mathcal{M})$$

is a covering map.

In particular, if the map $\Phi_{\mathcal{M}} : \ \mathcal{M} \to \Phi_{\mathcal{M}}(\mathcal{M})$ is generically injective, then $\Phi_{\mathcal{M}} : \ \mathcal{M} \to \Phi_{\mathcal{M}}(\mathcal{M})$ is a degree one covering map, which must be globally injective. □

Note that in many cases it is possible to find a subgroup Γ_0 of Γ, which is of finite index in Γ, such that its action on D is free and D/Γ_0 is smooth. In such cases we can consider the lift $\Phi_{\mathcal{M}_0} : \mathcal{M}_0 \to D/\Gamma_0$ of the period map $\Phi_{\mathcal{M}}$, with $\mathcal{M}_0$ a finite cover of $\mathcal{M}$. Then our argument can be applied to prove that $\Phi_{\mathcal{M}_0}$ is actually a covering map onto its image for polarized Calabi-Yau type manifolds with smooth moduli spaces.

In this subsection, we require that the moduli space $\mathcal{M}$ of Calabi-Yau type manifolds are smooth, and there are many nontrivial examples satisfying this requirement. As in the special cases of simply connected Calabi-Yau manifolds, one may see from Popp [14], Viehweg [22] and Szendroi [19] that the moduli spaces $\mathcal{M}$ of Calabi-Yau manifolds are smooth.

References

[1] A. Andreotti, On a theorem of Torelli. *Amer. J. Math.*, Vol. 80, No. 4 (1958), pp. 801–828.

[2] L. Auslander, and L. Markus, Holonomy of flat affinely connected manifolds. *Ann. Math.*, Second Series, Vol. 62, No. 1 (1955), pp. 139–151.

[3] S. Barannikov and M. Kontsevich Frobenius manifolds and formality of Lie algebras of polyvector fields. *Internat. Math. Res. Notices*, Vol. 1998, No. 4 (1998), pp. 201–215.

[4] S. Ferrara and J. Louis, Flat holomorphic connections and Picard-Fuchs identities from $N = 2$ supergravity. *Phys. Lett. B*, Vol 278, No 3 (1992), pp. 240–245.

[5] P. Griffiths, Periods of integrals on algebraic manifolds I, II Amer. J. Math. 90(1968), pp. 568-626, pp. 805-865.

[6] P. Griffiths, Periods of integrals on algebraic manifolds III Publ. Math. IHES. 38(1970), pp. 228-296.

[7] P. Griffiths, *Topics in transcendental algebraic geometry*, Volume 106 of *Annals of Mathematics Studies*, Princeton University Press, Princeton, NJ, (1984).

[8] F. Guan, K. Liu and A. Todorov, A global Torelli Theorem for Calabi-Yau manifolds, arXiv:1112.1163v2.

[9] A. Iliev and L. Manivel, Fano manifolds of Calabi-Yau type, preprint 2011

[10] S. Kobayashi, *Differential Geometry of Complex Vector Bundles*, Iwanami Shoten Publishers and Princeton University Press, (1987).

[11] S. Kobayashi and K. Nomizu, *Foundation of differential geometry I*, Wiley-Interscience, New York-London, (1963).

[12] K. Kodaira and J. Morrow, *Complex Manifolds*, AMS Chelsea Publishing, Porvindence, RI, (2006), Reprint of the 1971 edition with errata.

[13] Y. Matsushima, Affine structure on complex manifold. *Osaka J. Math.*, Vol 5, No. 2 (1986), pp. 215–222.

[14] H. Popp, *Moduli Theory and Classification Theory of Algebraic Varieties*, Volume 620 of *Lecture Notes in Mathemathics*, Springer-Verlag, Berlin-New York, (1977).

[15] I. I. Piatetski-Shapiro and I. R. Shafarevich, A Torelli theorem for algebraic surfaces of type K3. *USSR Izv. Ser. Math.*, Vol 5 (1971), pp. 547–588.

[16] W. Schmid, Variation of Hodge structure: the singularities of the period mapping. *Invent. Math.*, Vol. 22, No.3–4 (1973), pp. 211–319.

[17] W. Schmid, Letter to the authors. 2009.

[18] Y. Shimizu and K. Ueno, *Advances in moduli theory*, Volume 206 of *Translation of Mathematical Monographs*, American Mathematics Society, Providence, Rhode Island, (2002).

[19] B. Szendröi, Some finiteness results for Calabi-Yau threefolds. *Journal of the London Mathematical Society*, Second series, Vol. 60, No. 3 (1999), pp. 689–699.

[20] R. Torelli, Sulle Varieta' di Jacobi I, II. *Rendiconti R. Accad. ei Lincei*, Vol. 22, No.2 (1913), pp. 98–103, 437–441.

[21] M. Verbitsky, A global Torelli theorem for hyperkähler Manifolds. *Preprint arXiv:0908.4121 v6.*

[22] E. Viehweg, *Quasi-projective moduli for polarized manifolds*, Volume 30 of *Ergebnisse der Mathematik und ihrer Grenzgebiete (3) [Results in Mathematics and Related Areas (3)]*, Springer-Verlag, Berlin, (1995).

[23] C. Voisin, *Hodge theory and complex algebraic geometry I*, Cambridge Universigy Press, New York, (2002).

[24] C. Voisin, *Hodge theory and complex algebraic geometry II*, Cambridge Universigy Press, New York, (2002).

[25] A. Vitter, Affine structures on compact complex manifolds. *Invent. Math.*, Vol. 17 (1972), pp. 231–244.

[26] A. Weil, Sur le théorème de Torelli. *Séminaire N.Bourbaki*, Vol. 4 (1995), Exp. No. 151, pp. 207–211.

[27] A. Weil, Final Report on Contract 18(603)-57, *Scientific Works. Collected Papers*, vol. II (1958), pp. 390–395, 545–547.

Department of Mathematics,University of California at Los Angeles, Los Angeles, CA 90095-1555, USA
E-mail address: xjchen@math.ucla.edu

Department of Mathematics,University of California at Los Angeles, Los Angeles, CA 90095-1555, USA
E-mail address: fguan@math.ucla.edu

Center of Mathematical Sciences, Zhejiang University, Hangzhou, Zhejiang 310027, China; Department of Mathematics,University of California at Los Angeles, Los Angeles, CA 90095-1555, USA
E-mail address: liu@math.ucla.edu, liu@cms.zju.edu.cn